Linear Systems

R.J. Marks II Class Notes
Professor Thomas Krile
Rose-Hulman Institute of Technology (1970)

LINEAR SYSTEMS

	1 7:50	2 8:45	3 9:40	4 10:35	5 11:30	6 12:25	7 1:20	8 2:15	9 3:10
MON		LIN. ALG. A212		MECH II G222	PRINC. OF ECON A212	PHIL. A241	LIN. SYST. A206		
TUES		LIN ALG. A212				LINEAR SYST A206			
WED		LIN ALG A212		MECH II G222	PRINC OF ECON A212	PHIL A241			
THURS				MECH II G222	PRINC OF ECON A212	PHIL A241	LIN SYS A206	EE LAB I D101	
FRI		LIN ALG A212		MECH II G222	PRINC. OF ECON. A212	[redacted]	LIN SYST A206		

12-10-70

CHAPTERS 8 - END OF BOOK

WORK PRE-TEST FOR TOMORROW

JAN 7 → FIRST TEST (THURSDAY)

3 TESTS + FINAL

BELOW 50, FLUNK, CURVE OTHERWISE

12-11-70

TRANSFER FUNCTIONS, ASSUME ALL INITIAL TRANSFORMS = 0

KRAMER'S RULE

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

FIND I_2

$$I_2 = \frac{\begin{vmatrix} a_{11} & V_1 & a_{13} \\ a_{21} & V_2 & a_{23} \\ a_{31} & V_3 & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \Delta$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = A \Rightarrow |A| = a_{11}a_{22}a_{33} + a_{12}a_{22}a_{31} + a_{13}a_{21}a_{32}$$
$$- a_{31}a_{22}a_{13} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} + a_{21} \cdots$$

TRANSPOSE ⇒ ROWS TO COULOMBS
COLOMNS TO ROWS

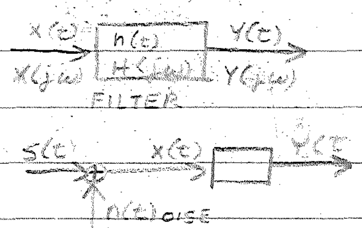
$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ 1 & 3 & 4 \end{bmatrix} \text{ COFACTORING} \Rightarrow \begin{bmatrix} 4 & -8 & 5 \\ -11 & 5 & -1 \\ 1 & -2 & -3 \end{bmatrix}$$

1-5-71

1-8-71

CHAPTER 9 - PROBABILITY

1-11-71



x(t) RARELY ANALYTICAL
∴ x(t) & Y(t) USUALLY DESCRIBED
PROBABALISTICALLY
(PROBABILITY MODEL OF x(t))
(MORE SPECIFICALLY STATISTICAL MODEL OF x(t))

TWO PROBLEM AREAS:

1) COMMUNICATIONS: FILTER OUT NOISE AS MUCH AS POSSIBLE
(MAXIMIZE $\frac{\text{SINGNAL ENERGY}}{\text{NOISE ENERGY}}$) i.e. DETECTION & SYNCRONIZATION

2) RADAR: DETECT & RANGING & RANGE RATE

CHAPTER NINE:

$P(A) = \frac{N_A}{N}$, FREQUENCY CONCEPT

FOR M POSSIBLE EVENTS WITH $P(i)$ FOR $i = 1, M$

$\Rightarrow \sum_{i=1}^{M} P(i) = 1$ IF THE EVENTS ARE MUTUALLY EXC.

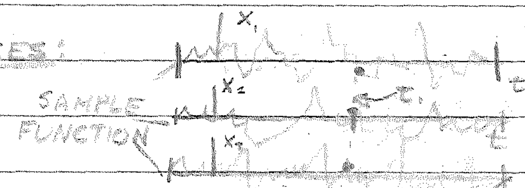
RELATION OF JOINT & CONDITIONAL PROBABILITY

$P(A,B) = P(A/B)P(B)$

$= P(B/A)P(A)$

JOINT PROB ; CONDITION; MARGINAL

RANDOM PROCESSES:



} COLLECTION OF RANDOM FUNCTIONS = RANDOM PROCESS

AND THE RANDOM PROCESS + PROB. LAW = ENSEMBLE

RANDOM VARIABLE x IS $x(t_1)$ FOR EACH RAND. FUNCTION

DISTRIBUTION AND DENSITY FUNCTIONS:

$P(x) = \text{PROB}\,(X \leq x)$

$0 \leq P(x) \leq 1$ NON-DECREASING

$\text{PROB}\,(x_1 \leq X \leq x_2) = P(x_2) - P(x_1)$

PROBABILITY DENSITY FUNCTION (P.D.F.)

$p(x)dx = \text{PROB}\,(x \leq X \leq x + dx) = 0$

$0 \leq p(x), \quad \int_{-\infty}^{x_1} p(x)\,dt = P(x_1) \cancel{= P(\infty)}$

AVERAGES & MOMENTS

$\bar{x} = E[x] = \int_{-\infty}^{\infty} x\,p(x)\,dx \Rightarrow \overline{f(x)} = \int_{-\infty}^{\infty} f(x)\,p(x)\,dx$

$\overline{x^2}$ = MEAN SQUARE VALUE

$\sqrt{\overline{x^2}}$ = R.M.S.

CENTRAL MOMENT: $E[x - \bar{x}] = 0$

$E[(x - \bar{x})^2] = \overline{x^2} - (\bar{x})^2 = \sigma^2$

1-12-71

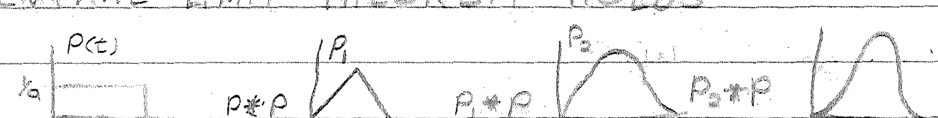
GAUSSIAN RANDOM VARIABLE:

1) CAN USE MATRIX FORM FOR MULTI-VARIABLE PROBLEMS

2) LINEAR COMBINATIONS OF G.R.V.'S ARE G.R.V.'s.

3) NEED 1^{st} & 2^{nd} MOMENTS ONLY TO COMPLETELY DESCRIBE G.R.V.'s

4) CENTRAL LIMIT THEOREM HOLDS



5) MATHEMATICALLY EASY TO WORK WITH

JOINT PROBABILITY FUNCTIONS

C.D.F. $0 \leq P(X,Y) \leq 1$

P.D.F. $p(x,y) = \frac{\partial^2 P(X,Y)}{\partial X \partial Y}$

$p_x(x) = \int_{-\infty}^{\infty} p(x,y)\,dy$ → MARGINAL DENSITY FUNCTION

$p_y(y) = \int_{-\infty}^{\infty} p(x,y)\,dx$

$E[XY] = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} xy\,p(x,y)\,dy$

$= E(X)E[Y]$ FOR STATISTICALLY INDEPENDENT R.V.'s

STATE INDEPENDENCE - IFF $p(x,y) = p(x)\,p(y)$, THEN

x & y ARE STATISTICALLY INDEPENDENT

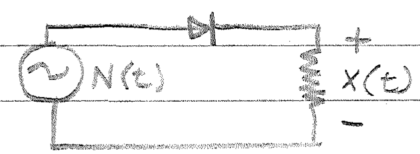
RANDOM PROCESSES (AS CONTRASTED WITH R.V.'s)

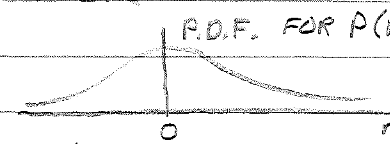
1) CONTINUOUS OR DISCRETE RANDOM PROCESSES

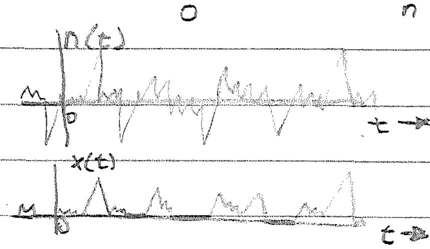
REFERS TO THE PROBABILITY LAW GOVERNING THE PROCESS & NOT THE SAMPLE FUNTIONS (THERE CAN BE NO δ FUNTIONS IN P.D.F.)

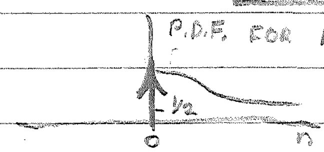
EX) GAUSS. NOISE AT RECTIFIER

LOOK AT OUT PUT









∴ PROCESS IS CONT. IF P.D.F. IS CONTINUOUS

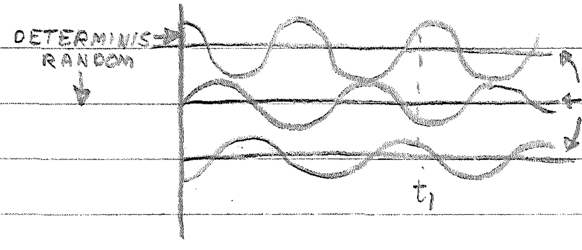
MIXED RANDOM PROCESS IF NOT.

2) DETERMINISTIC VS. NON-DETERMINISTIC

REFERS TO SAMPLE FUNCTION RATHER THAN PROBABILITY LAW.

EX) → DETERMINISTIC $A\cos(\omega t + \Theta)$ VARIES UNIFORMLY OVER 0–2π

CONST



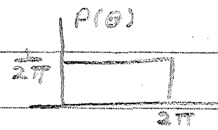
$x(t_1)$ IS A R.V.

3) STATIONARITY

$p[x(t_1)] = p[x(t_1+h)] \ni (t_1 \;\&\; t_1+h) \in (0,T)$, SIMILARLY

$p[x(t_1)x(t_2)] = p[x(t_1+h)\,x(t_2+h)]$ - MAY BE EXTENDED TO n^{TH} ORDER STATIONARITY

FROM PREVIOS EXAMPLE:



$$\mu = \int_0^{2\pi} A\cos(\omega t+\theta)\,\frac{1}{2\pi}\,d\theta = 0$$

PROBLEMS 8 & 9, CHAPT. 9

1-14-71

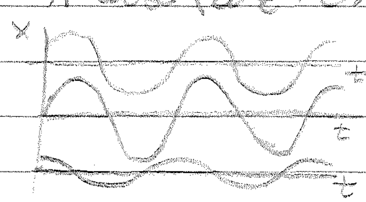
20.1 OR 20.2 → COMPUTOR PROGRAM → MONDAY

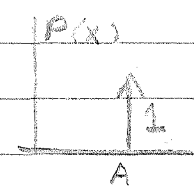
LECTURE:

4) ERGODIC PROCESS: STATIONARY PROCESS WHERE EACH SAMPLE FUNCTION HAS THE SAME STATISTICS AS THE ENSEMBLE

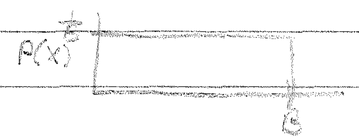
∴ TIME AVE = ENSEMBLE AVE

EX) $\overset{RV}{A}\cos(\omega t + \theta)^{RV}$ STATIONARY, BUT NOT ERGODIC

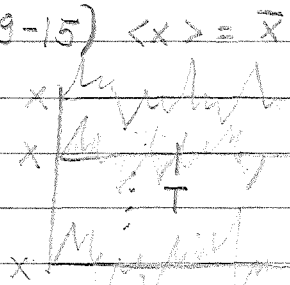




SAMPLE FUNCTION



9-15) $\langle x \rangle = \bar{x} \Rightarrow \lim_{t\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)\,dt = \int_{-\infty}^{\infty} x\,p(x)\,dx$



$$x_{AVE} = \frac{1}{T}\int_0^T x(t)\,dt$$

NO, IS NOT INTEGRATED OVER ALL TIME

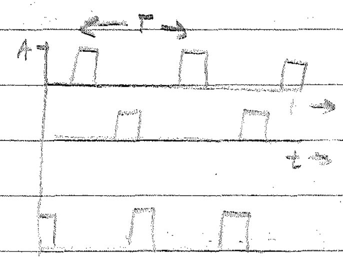
ERGODICITY CLAIMS AVE ($\bar{x}$) IS SAME AS $\lim_{T\to\infty} \frac{1}{T}\int_0^T x(t)\,dt$, OR SOMETHING LIKE THAT

(CONT.)

6

$$\overline{X} = \overline{X_{AVE}} = E[X_{AVE}]$$

9-14)

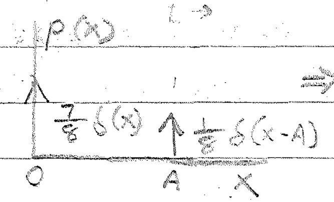


DISCRETE PROCESS (0 OR A)

A DETERMINISTIC PROCESS

STATIONARY PROCESS

ERGODIC? FIND $\langle x \rangle$, $\overline{X}$



$$\Rightarrow p(x) = \tfrac{7}{8}\delta(x) + \tfrac{1}{8}\delta(x-A)$$

$$\overline{X} = \int_{-\infty}^{\infty} x\, p(x) = \int_{-\infty}^{\infty} X\left(\tfrac{7}{8}\delta(x) + \tfrac{1}{8}\delta(x-A)\right)dx$$

$$= A/8$$

$$\langle x \rangle = \frac{1}{T}\int_0^T x(t)\,dt = \frac{1}{T}\int_0^{T/8} A\,dt = \frac{A}{8}$$

$$\Longrightarrow \langle x \rangle = \overline{X}$$

FIND σ^2

$$E[X] = \frac{A}{8}$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2\left[\tfrac{7}{8}\delta(x) + \tfrac{1}{8}\delta(x-A)\right]dx = \frac{A^2}{8}$$

$$\langle x^2 \rangle = \frac{A^2}{8} \qquad \sigma^2 = E[x^2] - E[x]^2$$

LOOKS GOOD FOR ERGOCITY

1-15-71

EXPECTED VALUE OF PRODUCT OF R.V. LEADS TO CORRELATION

$$E[x(t_1)\cdot x(t_2)] = E[x_1 x_2] = R_x(t_1, t_2) = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} x_1 x_2\, p(x_1, x_2)\, dx_2$$

IF PROCESS IS STATIONARY

$$R_x(t_1, t_2) = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} x_1 x_2\, p(x_1, x_2)\, dx_2$$

$$= \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} x_1(t+T)\, x_2(t+T)\, p[x_1(t+T)\, x_2(t+T)]\, dx_2$$

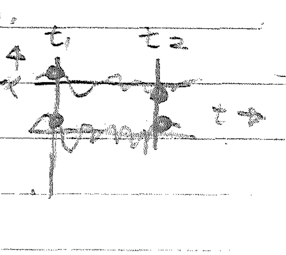
$$= R_x(t_1 + T, t_2 + T)$$

LET $T = -t' \Rightarrow R_x(0, t_2 - t_1)$; $x(t_1) = x_1$ & $x(t_2) = x_2$

$$R_x(0, t_2 - t_1) = R_x(t_2 - t_1)$$

$\Rightarrow$ NO MATTER WHERE ONE PLACES Δt IN TIME, SAME CORRELATION WILL BE EXPECTED.

CHANGING Δt ONLY CHANGES CORR FOR ENSEMBLE FOR STATIONARY PROCESS



$\therefore R_x$ IS FUNCTION OF ONE VARIABLE FOR STATIONARY PROCESS, WHICH IS $t_2 - t_1$

TIME AUTOCORRELATION:

$$\mathcal{R}_x(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x_1(t)\,x_2(t+\tau)\,dt$$

$$= R_x \text{ FOR ERGODIC PROCESS}$$

NOTE $R_x(0) = E[x^2] = \overline{X^2}$ = MEAN SQUARE VALUE OF X

SIMILARITY FUNCTION (R_x) SHOWS HOW SIMILAR 2 R.V. ARE

LET $Y(t) = X(t) - \rho X(t+\tau)$

$E[Y^2]$ IS A MEASURE OF CLOSENESS OF $x(t)$ TO $x(t+\tau)$

X IS A ZERO-MEAN RANDOM PROCESS

$$E[Y^2] = E\left[(x(t) - \rho x(t+\tau))\right]^2$$

$$= E\left[x^2(t) - 2\rho x(t)\,x(t+\tau) = \rho^2 x^2(t+\tau)\right]$$

$$\Rightarrow \overline{Y^2} = \overline{X^2} - 2\rho R_x(\tau) + \rho^2\,\overline{x^2}$$

$$\sigma_Y^2 = \overline{Y^2} - \overline{Y}^2$$

$$\Rightarrow \sigma_Y^2 + \overline{Y}^2 = \sigma_x^2 + \overline{X}^2 - 2\rho R_x(\tau) + \rho^2\left[\sigma_x^2 + \overline{X}^2\right]$$

FOR ZERO MEAN RANDOM PROCESS

$$\overline{Y}^2 = \overline{X}^2 = 0$$

$$\Rightarrow \sigma_Y^2 = \sigma_x^2 - 2\rho R_x(\tau) + \rho^2\sigma_x^2$$

$$d\sigma_Y^2/d\rho = 0 - 2R_x(\tau) + 2\rho\sigma_x^2 = 0 \Rightarrow \rho = \frac{R_x(\tau)}{\sigma^2_x} \text{ FOR MINIMUM}$$

$$\Rightarrow \begin{cases} R_x(\tau) \to \text{SIMILARITY PARAMETER} \\ \sigma^2_x \to \text{UNCERTAINTY PARAMETER} \end{cases}$$

PROVE:

$-1 \leq \rho < 1$

NOTE $\rho = \dfrac{E[x(t)\,x(t+\tau)]}{E[x^2(t)]}$ → FOR ZERO MEAN PROCESS

CONSIDER $E\left[(x(t) \pm x(t+\tau))^2\right]$

$$= E\left[x^2(t) \pm 2x(t)x(t+\tau) + x^2(t+\tau)\right]$$

$$= \sigma_x^2 + R_x(\tau) + \sigma_x^2 = 2\sigma_x^2 \pm 2R_x(\tau) \geq 0$$

$$\Rightarrow 1 \pm R_x(\tau)/\sigma_x^2 \geq 0$$

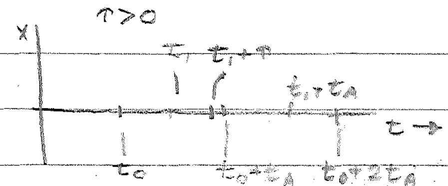
$$\Rightarrow -1 \leq \rho \leq 1$$

$\rho=1 \Rightarrow$ 2 FUNCTIONS HIGHLY CORRELATED

$\rho=-1 \Rightarrow$ " " " 180° OUT OF PHASE

$\rho=0 \Rightarrow$ NO CORRELATIONS

1-18-70



$\bar{X}=0$

WISH TO FIND $R_x(\tau) \triangleq E[X(t_1)X(t_1+\tau)]$

FOR $|\tau|>t_A$; t_1 & $t_1+\tau$ ARE IN DIFF. INTERVAL

$\therefore$ $X(t_1)$ & $X(t_1+\tau)$ ARE INDEPENDENT

AND $R_x(\tau)=0=E[X(t_1)]E[X(t_1+\tau)]=\bar{X}\bar{X}=0$

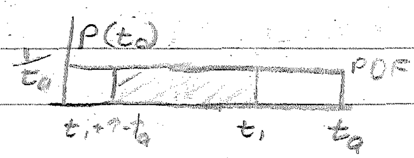
FOR $0 \le |\tau| \le t_A$

$|\tau| \le t_A$ & $|\tau| \ge 0$

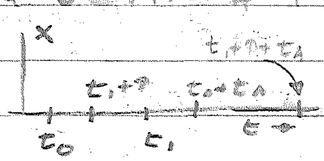
WANT PROB (t_1 & $t_1+\tau$ ARE IN THE SAME INTERVAL)

$= \text{PROB}(t_1+\tau < t_0+t_A < t_1+t_A)$

$= \text{PROB}(t_1+\tau-t_A < t_0 \le t_1) = \dfrac{t_1-(t_1+\tau-t_a)}{t_a} = \dfrac{t_a-\tau}{t_a}$



FOR $0 \le |\tau| \le t_A$ & $\tau<0$



PROB{$X(t_1)$ & $X(t_1+\tau)$ ARE IN SAME INT}

$= \text{PROB}(t_1 < t_0+t_A \le t_1+\tau+t_A)$

$= \text{PROB}(t_1-t_0 < t_0 \le t_1+\tau)$

$= \int_{t_1-t_a}^{t_1+\tau} P(t_0)\,dt = \dfrac{t_a-\tau}{t_a}$

USING ABOVE $P(t_0)$ PDF

PROB $[X(t_1)$ & $X(t_1+\tau)$ ARE IN SAME INT$] = \dfrac{t_a-|\tau|}{t_a}$

$$R_x(\tau) = \int_{\text{ALL } X(t_1)} dx(t_1) \int_{\text{ALL } X(t_1+\tau)} x(t_1)\,x(t_1+\tau)\,P[x(t_1)\,x(t_1+\tau)]\,dx(t_1+\tau)$$

ALL $x_1(t)$ & $x_1(t+\tau) \ni$ THEY ARE IN SAME INTERVAL

$$= \int dx(t_1) \int x(t_1)x(t_1+\tau)\,P[x(t_1)\,x(t_1+\tau)]\,dx(t_1+\tau)$$

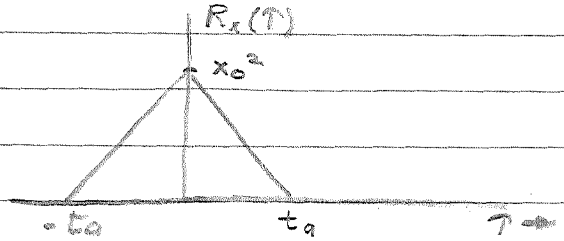
$$+ \int dx(t_1) \int x(t_1)x(t_1+\tau)\,P[x(t_1)\,x(t_1+\tau)]\,dx(t_1+\tau)$$

ALL $x(t)$ & $x(t_1+\tau)$ $\ni$ THEY ARE NOT IN SAME INTER $=0$

(CONT)

$$R_x(\tau) = x_0^2 \int dx(t_1) \int p(\cdot,\cdot)\, dx(t_1+\tau) = x^2\left(\frac{t_a-|\tau|}{t_a}\right)$$

OVER ALL $x(t_1)$ & $x(t_1+\tau)$
$\Rightarrow$ THEY ARE IN SAME INTERVAL



PROPERTIES OF $R_X(\tau)$

1) $R_x(0) = \overline{X^2}$ = MEAN SQUARE VALUE OF X
2) $R_x(\tau) = R_x(-\tau) \Rightarrow R_x(\tau)$ IS AN EVEN FUNCTION
3) $|R_x(\tau)| \leq R_x(0) \Rightarrow$ SHOWN BY $E[(x(t)+x(t+\tau))^2] \geq 0$
4) $\overline{X}$ = MEAN VALUE MAY BE FOUND BY $\lim_{|\tau|\to\infty} \sqrt{R_x(\tau)}$

BECAUSE A D.C. (AVERAGE) COMPONENT APPEARS IN $R_x(\tau)$, $\lim_{|\tau|\to\infty} R_x(\tau) = 0$ IF PROCESS IS ZERO MEAN AND NON PERIODIC

EX) $x(t) = x_0 + n(t)$; $x(t+\tau) = x_0 + n(t+\tau)$

$$R_x(\tau) = E\left[x_0^2 + 2x_0 n(t) + 2x_0 n(t+\tau) + n(t)\, n(t+\tau)\right]$$
$$= x_0^2 + 2x_0\overline{n} + 2x_0\overline{n} + R_n(\tau)$$

1-19-70

CROSS CORRELATION FUNCTIONS

FOR $x(t)$ & $y(t)$, ASSUMING STATIONARITY

$$R_{xy}(\tau) = E\left[x(t_1)\, y(t_1+\tau)\right]$$
$$= \int_{-\infty}^{\infty} dx(t_1) \int_{-\infty}^{\infty} x(t_1)\, y(t_1+\tau)\, p\left[x(t_1), y(t_1+\tau)\right] dy(t_1+\tau)$$

LET $t_2 = t_1 + \tau$

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} dx(t_2-\tau) \int_{-\infty}^{\infty} x(t_2-\tau)\, y(t_2)\, p\left[x(t_2-\tau)\, y(t_2)\right] dy(t_2)$$
$$= \int_{-\infty}^{\infty} dy(t_2) \int_{-\infty}^{\infty} y(t_2)\, x(t_2-\tau)\, p\left[y(t_2)\, x(t_2-\tau)\right] dx(t_2-\tau)$$

$$\Rightarrow R_{xy}(\tau) = R_{yx}(-\tau)$$

$\mathcal{R}_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t_1)\, y(t_1+\tau)\, dt_1$ = TIME AVERAGE OUTCOME

FOR ERGODIC PROCESS:

$$\mathcal{R}_{xy}(\tau) = R_{xy}(\tau)$$

PROPERTIES OF CROSS-CORRELATION FUNCTIONS

1) $R_{XY}(0) = R_{YX}(0)$ & $R_{XY}(0)$ & $R_{YX}(0)$ HAVE NO PHYSICAL MEANING

2) NOT NECESSARILY SYMMETRIC, BUT $R_{XY}(\tau) = R_{YX}(-\tau)$

3) $|R_{XY}(\tau)| \leq \sqrt{R_X(0)}\sqrt{R_Y(0)}$

PROOF: CONSIDER $E\left[\left(\frac{X}{\sqrt{R_X(0)}} \pm \frac{Y}{\sqrt{R_Y(0)}}\right)^2\right] \geq 0$

$$\Rightarrow E[\ \ ] = E\left[\frac{X^2}{R_X(0)} \pm \frac{2XY}{\sqrt{R_X(0)R_Y(0)}} + \frac{Y^2}{R_Y(0)}\right]$$

$$= \left[\frac{R_X(0)}{R_X(0)} \pm \frac{2R_{XY}(\tau)}{\sqrt{R_X(0)R_Y(0)}} + \frac{R_Y(0)}{R_Y(0)}\right] = 2\left[1 \pm \frac{R_{XY}(\tau)}{\sqrt{R_X(0)}\sqrt{R_Y(0)}}\right]$$

$$\Rightarrow |R_{XY}(\tau)| \leq \sqrt{R_X(0)}\sqrt{R_Y(0)}$$

4) IF X & Y ARE TWO STATISTICALLY INDEPENDENT PROCESSES, AND ONE HAS ZERO MEAN, THEN $R_{XY}(\tau) = R_{YX}(\tau) = 0$

CONVERSE TRUE IF BOTH X & Y ARE GAUSSIAN RANDOM VARIABLES

SUMS OF RANDOM PROCESSES:

1) AUTOCORRELATION OF SUM = SUM OF AUTOCORRELATIONS + ALL CROSS CORRELATIONS

EX) $W = X + Y + Z$

$$R_{X+Y+Z} = R_W = R_X + R_Y + R_Z + R_{XY} + R_{XZ} + R_{YX} + R_{YZ} + R_{ZX} + R_{ZY}$$

IF STATISTICAL INDEPENDENT: $R_{XY} = R_{YX}$ ETC.

IF ALL ZERO MEAN: $R_{XY} = R_{ZX} = \ldots = 0$

VECTORS:

$$\underset{\sim}{X}(t) = \begin{bmatrix} X_1(t) \\ \vdots \\ X_n(t) \end{bmatrix}$$ ∋ $X_i(t)$ ARE RANDOM FUNCTIONS FROM (MAYBE DIFFERENT) RANDOM PROCESSES

$$R_X(t_1, t_2) = E\left[\underset{\sim}{X}(t_1)\, X^T(t_2)\right]$$

$$= \begin{bmatrix} R_{11}(t_1, t_2) & \cdots & R_{1m}(t_1, t_2) \\ \vdots & & \vdots \\ R_{m1}(t_1, t_2) & \cdots & R_{mm}(t_1, t_2) \end{bmatrix}$$ AUTOCORR-MATRIX

CROSS CORR.

SUPPOSE $\underset{\sim}{x}(t)$ & $\underset{\sim}{y}(t)$

$$R_{XY}(\tau) = E\left[x(t)\, Y^T(t+\tau)\right]$$

1-21-70

CORRELATION MATRIX OF SAMPLED FUNCTIONS

$$\underset{\sim}{x} = \begin{bmatrix} x(t_1) \\ x(t_2) \\ \vdots \\ x(t_n) \end{bmatrix}$$

ALL $X(t_i)$ ARE RANDOM VARIABLES, SINGULAR NUMBER'S, & ALL FROM SAME SAMPLE FUNCTION

$$E\left[\underset{\sim}{x}\,\underset{\sim}{x}^T\right] = \begin{bmatrix} R_x(t_1 t_1) & R_x(t_1 t_2) & \cdots & R_x(t_1, t_n) \\ R_x(t_2 t_1) & & & \vdots \\ R_x(t_n t_1) & \cdots & & R_x(t_n t_n) \end{bmatrix}$$

ALL $R_x(t_i\, t_i) = R_x(0)$ ASSUMING STATIONARITY

IF THE PROCESS IS ZERO-MEAN: $R_x(0) = \sigma_x^2$

FOR UNIFORM SAMPLING, $R_x(t_i, t_j) = R_x((j-i)\Delta t)$ $\rightarrow = R_x((i-j)\Delta t)$

$$\Rightarrow \underset{\sim}{R}_x = E\left[\underset{\sim}{x}\,\underset{\sim}{x}^T\right] = \underbrace{\begin{bmatrix} R_x(0) & R_x(\Delta t) & \cdots & R_x((n-1)\Delta t) \\ R_x(\Delta t) & R_x(0) & & \\ & R_x(\Delta t) & \ddots & \\ R[(n-1)\Delta t] & \cdots & R_x(\Delta t) & R_x(0) \end{bmatrix}}_{\text{SYMETRIC}}$$

$\Rightarrow n$ DIFFERENT MEMBERS

DEFINE COVARIANCE: $E\left[(x_1 - \bar{x}_1)(x_2 - \bar{x}_2)\right]$

$$= E\left[x_1 x_2 - x_1 \bar{x}_2 - \bar{x}_1 x_2 + \bar{x}_1 \bar{x}_2\right] = \sqrt{\sigma_1^2}\sqrt{\sigma_2^2}\,\rho_{12} = \sigma_{12}^2$$

$$= E\left[x_1 x_2\right] - \bar{x}_1 \bar{x}_2 = \overline{x_1 x_2}$$

CORRELATION COEFFICIENT

COVARIANCE MATRIX $\underset{\sim}{\Lambda}_x$

$$\underset{\sim}{\Lambda}_x = E\left[(\underset{\sim}{x} - \underset{\sim}{\bar{x}})(\underset{\sim}{x}^T - \bar{x}^T)\right]$$

$$\underset{\sim}{\Lambda} = E\left[\begin{bmatrix} x_1 - \bar{x}_1 \\ x_2 - \bar{x}_2 \\ \vdots \\ x_n - \bar{x}_n \end{bmatrix}\begin{bmatrix} x_1 - \bar{x}_1 & x_2 - \bar{x}_2 & \cdots & x_n - \bar{x}_n \end{bmatrix}\right] = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho_{12} & \sigma_1\sigma_3\rho_{13} & \cdots & \sigma_1\sigma_n\rho_{1n} \\ \sigma_2\sigma_1\rho_{12} & & \text{SAME} & & \text{SAME} \\ \vdots & & & & \\ \sigma_n\sigma_1\rho_{n1} & & \cdots & & \sigma_n^2 \end{bmatrix}$$

$\rho_{ij} = \rho_{ji}$; $\rho_{ii} = 1$; AGAIN SYMETRIC

$$\Rightarrow \underset{\sim}{\Lambda}_x = \sigma^2 \begin{bmatrix} 1 & \rho_{12} & \rho_{13} & \cdots & \rho_{1n} \\ \rho_{21} & 1 & \rho_{23} & & \vdots \\ & & & \ddots & \vdots \\ \rho_{n1} & & \cdots & & 1 \end{bmatrix}$$

CONSIDER $\underset{\sim}{\mu} = \begin{bmatrix} \frac{x_1-\bar{x}_1}{\sigma_1} \\ \frac{x_2-\bar{x}_2}{\sigma_2} \\ \vdots \\ \frac{x_n-\bar{x}_n}{\sigma_n} \end{bmatrix}$ ASSUME THE x_i'S ARE <u>NOT</u> ALL FROM THE SAME RANDOM PROCESS

LOOK AT $E[\underset{\sim}{\mu}\underset{\sim}{\mu}^T]$

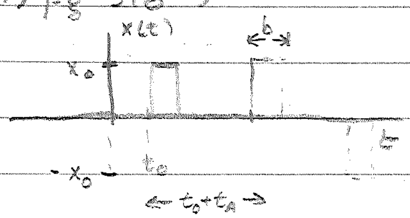
$$E[\underset{\sim}{\mu}\underset{\sim}{\mu}^T] = E\begin{bmatrix} \frac{(x_1-\bar{x}_1)^2}{\sigma_1^2} & \frac{(x_1-\bar{x}_1)(x_2-\bar{x}_2)}{\sigma_2\sigma_1} & \cdots & \frac{(x_1-\bar{x}_1)(x_n-\bar{x}_n)}{\sigma_n\sigma_1} \\ \vdots & & & \\ \frac{(x_n-\bar{x}_n)(x_1-\bar{x}_1)}{\sigma_n\sigma_1} & - & - & \frac{(x_n-\bar{x}_n)^2}{\sigma_n^2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \rho_{12} & \rho_{13} \cdots & \rho_{1n} \\ \rho_{21} & 1 & & \vdots \\ \vdots & & \ddots & \\ \rho_{2n} & - & - & 1 \end{bmatrix}$$

$\Rightarrow$ MUST NORMALIZE ALL x_i'S ARE NOT FROM SAME RANDOM PROCESS

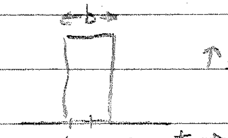
1-23-71

#10-1) pg 318-9



t_0 IS UNIFORM R.V. BETWEEN 0 & t_A

a) $E[x(t)x(t+\tau)]$ $\tau \le b$



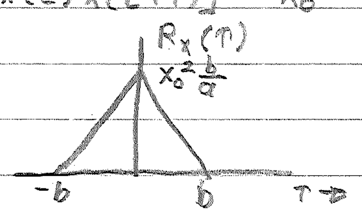
PROB $[t$ & $t+\tau \in b$, AND $t \in b]$

$= \text{PROB}[t$ & $t+\tau \in b / t \in b]\ \text{PROB}[t \in b]$

FROM PREVIOUS EXAMPLE $= \left[\frac{b-|\tau|}{b}\right]\left[\frac{b}{t_a}\right] = \frac{b-|\tau|}{t_a}$

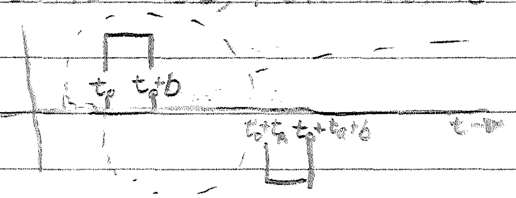
ELSEWHERE $E[x(t)x(t+\tau)] = E(0) = 0$

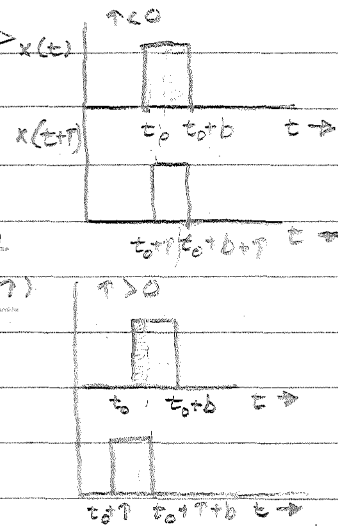
$\Rightarrow E[x(t)x(t+\tau)] = x_0^2 \left[\frac{b-|\tau|}{t_a}\right] + 0 = R_x(\tau)$



b) $R_x(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)\,x(t+\tau)\,dt$

SPECIAL CASE, FOR $x(t)$ IS PERIODIC





SAME AS CONVOLUTION WITHOUT FLIPPING

$\tau<0$ $\quad R_{x_1}(\tau) = \frac{1}{t_a}\int_{t_0+\tau}^{t_0+b} x_0^2\,dt = \frac{x_0^2(b-\tau)}{t_a}$

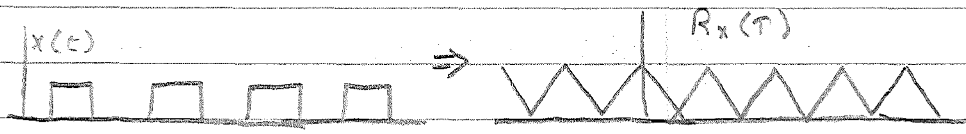
$\tau>0$ $\quad R_{x_2}(\tau) = \frac{1}{t_a}\int_{t_0}^{t_0+\tau+b} x_0^2\,dt = \frac{x_0^2(b-\tau)}{t_a}$

$\Rightarrow R_x(\tau) = \frac{x_0^2}{t_a}(b-|\tau|)$ FOR $\tau \le b$

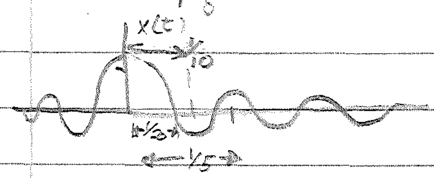
(SAME AS PREVIOUS EXAMPLE)

c) POSSIBLY, FOR $R_x(\tau) = R_x(\tau)$

d) $\overline{x^2} = R_x(0) = \frac{b x_0^2}{t_a}$



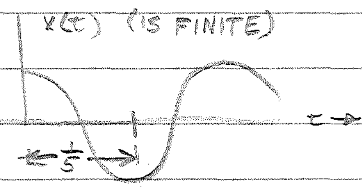
10-2 pg 319



$R_x(\tau) = 100\,e^{-|\tau|}\cos 10\pi\tau$

a) $\overline{X^2} = R_x(0) = 100$

b) $\tau = \frac{1}{5}$



c) UNCORRELATED

d) $\rho \doteq -1$

e) 0 ; $\lim_{\tau\to\infty} R_x(\tau) = \overline{X}^2 = 0$

1-25-71

TEST 2-5-71: NO CLASSES THURS. & FRI.

LECTURE

CAN'T TAKE FOURIER TRANSFORM OF RANDOM PROCESS

FOR $\int_{-\infty}^{\infty} x(t)\,dt \not< \infty$

CAN TAKE LAPLACE (TWO SIDED)

SOLUTION: USE TRUNCATED VERSION $X_T(t)$ -

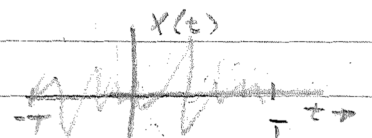
FOR $-\infty < -T \le T < \infty$

$\left(\overline{x} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\,dt\right) \Rightarrow$ THEN: DEFINE

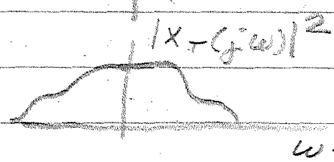
$X_T(j\omega) = \mathcal{F}\{X_T(t)\} = \int_{-\infty}^{\infty} x_T(t)\, e^{-j\omega t}\,dt$. THIS EXISTS $\forall\ T<\infty$

FROM PARSEVAL'S THEOREM:

$$\int_{-\infty}^{\infty} x_T^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X_T(j\omega)|^2\,d\omega$$



$\int_{-\infty}^{\infty} x_T^2(t)\,dt$ = TOTAL ENERGY IN $X_T(t)$



$|X_T(j\omega)|^2 \Rightarrow \frac{\text{POWER}}{\text{UNIT BANDWIDTH}}$

NOW $\frac{1}{2T}\int_{-\infty}^{\infty} x_T^2(t)\,dt = \frac{1}{4\pi T}\int_{-\infty}^{\infty} |X_T(j\omega)|^2\,d\omega$

$\Rightarrow$ EQUATE AVERAGE POWER

TAKE $E\left[\frac{1}{2T}\int_{-\infty}^{\infty} x_T^2(t)\,dt\right] = E\left[\frac{1}{4\pi T}\int_{-\infty}^{\infty} |X_T(j\omega)|^2\,d\omega\right]$

TAKE $E[\cdot]$ INSIDE INTEGRAL, TAKE LIMITS AS $T\to\infty$

$$\Rightarrow \lim_{T\to\infty} \frac{1}{2T}\left[\int_{-\infty}^{\infty} \underbrace{E[x_T^2(t)]}_{\overline{x^2}}\,dt\right] = \lim_{T\to\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{E[|X_T(j\omega)|^2]}{2T}\,d\omega\right]$$

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-\infty}^{\infty} \overline{x^2}\,dt$$

$$\Rightarrow \overline{x^2} = \lim_{T\to\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{E[|X(j\omega)|^2]}{2T}\,d\omega\right]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{E[|X(j\omega)|^2]}{2T}\,d\omega$$

$$= \frac{1}{2\pi} S_x(\omega) \ni S_x(\omega) = \text{POWER SPECTRAL DENSITY}$$

$\Rightarrow$ BY DEF; $S_x(\omega) = \lim_{T\to\infty} \frac{E[|X_T(j\omega)|^2]}{2T}$ $\frac{\text{POWER}}{\text{UNIT FREQ.}}$

$$\overline{x^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)\,d\omega$$

Pg 342

11-1) $S_x(\omega)$ IS POSITIVE, REAL, & EVEN

a) ONE SIDED SPECTRAL DENSITY $G_x(f)$

$$\overline{X^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{E[|X(j\omega)|^2]}{2T}\, d\omega$$

$$= \frac{1}{\pi}\int_{0}^{\infty} \lim_{T\to\infty} \frac{E[|X(j\omega)|^2]}{2T}\, d\omega$$

$$\omega = 2\pi f \Rightarrow d\omega = 2\pi\, df$$

$$\Rightarrow \overline{X^2} = \frac{1}{\pi}\int_{0}^{\infty} \lim_{T\to\infty} \frac{E[|X(2\pi f)|^2]}{2T}\, 2\pi\, df$$

$$= \int_{0}^{\infty} \lim_{T\to\infty} \frac{E[|X(2\pi f)|^2]}{T}\, df$$

$$\Rightarrow G_x f = \frac{E[|X(2\pi f)|]^2}{T}$$

b) FACTOR OF 2

c) $S_x(\omega) = \frac{10\omega^2}{\omega^4 + 2\omega^2 + 4} \Rightarrow G_x(f) = \frac{20(2\pi f)^2}{(2\pi f)^4 + 2(2\pi f)^2 + 4}$

$-\infty < \omega < \infty$ $\qquad 0 \le \omega < \infty$

2-1-71

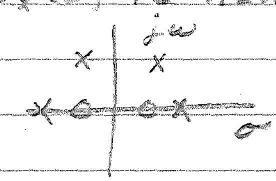
SPECTRAL DENSITIES AS FUNCTION OF S

$S_x(\omega) \to S_x(S)$; REPLACE ω BY $-jS$ OR ω^2 BY $-S^2$

(SPECTRAL DENSITY FOURIER TRANSFORM OF AUTOCORR FUNC.)

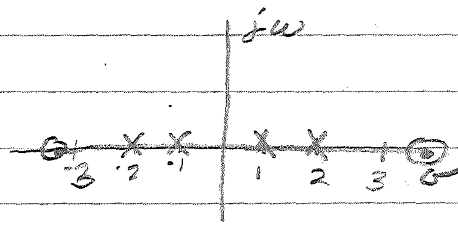
NOTE $S_x(S)$ IS ALWAYS SYMMETRIC ABOUT $j\omega$ AXIS

EX)



EX) $S_x(\omega) = \frac{\omega^2 + 10}{\omega^4 + 5\omega^2 + 4} \Rightarrow S_x(s) = \frac{-s^2 + 10}{s^4 - 5s^2 + 4}$

$$S_x(S) = \frac{(S+\sqrt{10})(S+\sqrt{10})}{(S^2-4)(S^2-1)} = \frac{(S+\sqrt{10})(-S+\sqrt{10})}{(S+2)(S-2)(S+1)(S-1)}$$



REALIZE:

$$S_x(\omega) = \mathcal{F}\{R_x(\tau)\}$$

$$\Rightarrow R_x(\tau) = \mathcal{F}^{-1}\{S_x(\omega)\}$$

$$\therefore R_x(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega) e^{j\omega\tau} d\omega$$

$$R_x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)\, d\omega = \overline{X^2}$$

TAKE $S_x(s)$ & PERFORM INVERSE $\mathcal{L}$

$$\Rightarrow R_x(0) = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} S_x(s)\,ds = \overline{x^2}$$

(NOTE 2-SIDED LAPLACE)

TWO SCHEMES FOR $R_x(0)$

① $S_x(s) = \dfrac{C(s)\,C(-s)}{D(s)\,D(-s)} = \dfrac{(s+\sqrt{10})(-s+\sqrt{10})}{[(s+1)(s+2)][(s-1)(s-2)]}$ ← FROM PREVIOUS EXAMPLE

$$= \frac{(s+\sqrt{10})(-s+\sqrt{10})}{(s^2+3s+2)(s^2-3s+2)}$$

$C(s) = C_{n-1}s^{n-1} + C_{n-2}s^{n-2} + \dots + C_0$

$D(s) = D_n s^n + D_{n-1}s^{n-1} + \dots + D_0$

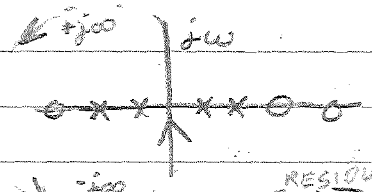
US TABLE IN BOOK. FOR THIS EXAMPLE $n=2$ (HIGHEST POWER)

$I_2 = R_x(0) = \dfrac{c_1^2 d_0 + c_0^2 d_2}{2 d_0 d_1 d_2}$ FROM TABLE 11-1

$$= \left[(1)^2(2) + (10)(1)\right] / 2(2)(3)(1) = 1 = \overline{x^2}$$

② $R_x(0) = \dfrac{1}{2\pi j}\displaystyle\int_{-j\infty}^{j\infty} S_x(s)\,ds$

$= \sum$ RESIDUES AT POLES ENCLOSED BY THE CURVE IF $C_n < D_n$



RESIDUES

$$S_x(s) = \frac{A}{s+2} + \frac{B}{s+1} + \frac{C}{s-2} + \frac{D}{s-1}$$

$$\Rightarrow R_x(0) = A + B = -\frac{1}{2} + \frac{3}{2} = 1 = \overline{x^2}$$

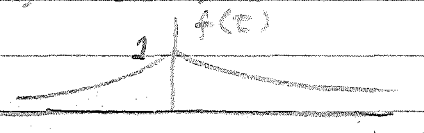
TWO-SIDED $\mathcal{L}$: $F_{II}(s) = \int_{-\infty}^{\infty} f(t)\,e^{-st}\,dt$

1) REGION OF CONVERGENCE IS IMPORTANT

2) TO DETERMINE THE CONVERGENT REGION LOOK AT $t \geq 0$ & $t \leq 0$

$$\Rightarrow F_{II}(s) = \int_{-\infty}^{0} f(t)\,e^{-st}\,dt + \int_{0}^{\infty} f(t)\,e^{-st}\,dt$$

Ex) $f(t) = e^{-a|t|}$

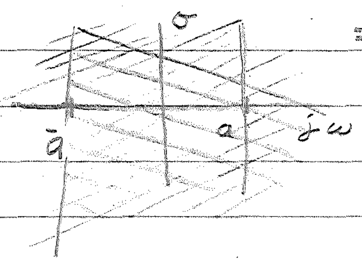


$F_{II}(s) = \int_{-\infty}^{0} e^{at} e^{-st}\,dt + \int_{0}^{\infty} e^{-at} e^{-st}\,dt$

1) $e^{at} e^{-\sigma t} e^{-j\omega t} = e^{(a-\sigma)t} e^{-j\omega t}$

$\Rightarrow a - \sigma > 0;\ a > 0$

2) $e^{(-a-\sigma)t} \Rightarrow -a - \sigma < 0 \Rightarrow a > -\sigma$



$\Rightarrow f(t)$ CONVERGES $-a < \sigma < a$

$\therefore F_{II}(s) = \frac{-1}{s-a} + \frac{1}{s+a} = \frac{-2a}{s^2 - a^2}$

$= S_x(s)$

$\Rightarrow S_x(\omega) = \frac{2a}{\omega^2 + a^2}$

BY CONVENTION: EXPAND $S_x(s)$ IN PARTIAL FRACTION FORM & LET ALL LEFT HAND POLES RESULT IN $t \geq 0$ $f(t)$'s & LET ALL RIGHT HAND POLE TERMS RESULT IN $t \leq 0$ $f(t)$'s.

FOR RIGHT HAND POLE TERM:

1. REPLACE s BY $(-s)$
2. TAKE ONE SIDED $\mathcal{L}^{-1}$
3. REPLACE t BY $-t$

1-2-71

TO TAKE $\mathcal{L}$ OF $f(t)$ FOR $t < 0$

1. REPLACE t BY $-t$
2. TAKE ONE-SIDED LAPLACE
3. REPLACE s BY $-s$

Ex) $S_x(s) = \frac{(s+\sqrt{10})(-s+\sqrt{10})}{(s+1)(s+2)(s-2)(s-1)}$ WHERE $S_x(\omega) = \frac{\omega^2+10}{\omega^4+5\omega^2+4}$

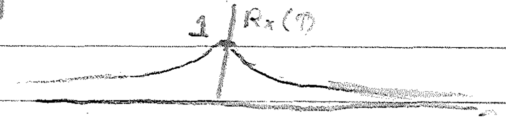
$R_x(\tau) = \mathcal{F}^{-1}\{S_x(\omega)\} = F_{II}^{-1}\{S_x(s)\}$

$S_x(s) = \frac{A}{s+2} + \frac{B}{s+1} + \frac{C}{s-2} + \frac{D}{s-1}$

$= \frac{-1/2}{s+2} + \frac{3/2}{s+1} + \frac{1/2}{s-2} - \frac{3/2}{s-1}$

$\mathcal{L}^{-1}\{S_x(s)\} = R_x(\tau) = \underbrace{-\tfrac{1}{2}e^{-2\tau} + \tfrac{3}{2}e^{-\tau}}_{\tau \geq 0}\ \underbrace{-\tfrac{1}{2}e^{2\tau} + \tfrac{3}{2}e^{\tau}}_{\tau \leq 0}$

$= -\frac{1}{2}e^{-2|\tau|} + \frac{3}{2}e^{-|\tau|}$



11-5) (Pg 343)

$$S_x(\omega) = \frac{\omega^2}{\omega^6 + 1}$$

$$S_x(s) = \frac{-s^2}{-s^6+1} = \frac{s^2}{s^6-1} = \frac{s^2}{\underbrace{(s+1)}_{LHP}\underbrace{(s^2-s+1)(s-1)}_{RHP}\underbrace{(s^2+s+1)}_{LHP}}$$

$C(s) = s$

$D(s) = (s+1)(s^2+s+1) = s^3+2s^2+2s+1$

FROM TABLE 11-1 Pg 335

$$R_x(0) = \overline{x^2} = I_3 = \frac{c_2^2 d_0 d_1 + (c_1^2 - 2c_0c_2)d_0d_3 + c_0^2 d_2 d_3}{2d_0d_3(d_1d_2 - d_0d_3)} = \frac{1}{6}$$

11-6) $S_x(s) = \dfrac{s^2}{(s+1)(s-1)(s^2-s+1)(s^2+s+1)}$

$$= \frac{A}{s+1} + \frac{B}{s-1} + \frac{Cs+D}{s^2+s+1} + \frac{Es+F}{s^2-s+1}$$

$$= \frac{-1/6}{s+1} + \frac{1/6}{s-1} + \frac{\frac{1}{3}s+1/6}{s^2+s+1} + \frac{-\frac{1}{3}s+1/6}{s^2-s+1}$$

$$= \underbrace{\frac{-1/6}{s+1} + \frac{\frac{1}{3}(s+\frac{1}{2})}{(s+1/2)^2+3/4}}_{\tau > 0} + \frac{1/6}{s-1} + \frac{-1/3\, s + \frac{1}{6}}{s^2-s+1}$$

$$\mathcal{L}\{S_x(s)\}(\tau > 0) = -\frac{1}{6}e^{-\tau} + \frac{1}{3}e^{\tau}\cos\frac{\sqrt{3}}{2}\tau$$

$$\Rightarrow \mathcal{L}\{S_x(s)\} = R_x(\tau) = -\frac{1}{6}e^{-|\tau|} + \frac{1}{3}e^{|\tau|}\cos\frac{\sqrt{3}}{2}\tau$$

WHITE NOISE:

$R_x(\tau) = S_0\delta(t)$; $S_x(\omega) = S_0$

$X \rightarrow \boxed{h(t)} \rightarrow Y$

$$S_{xy}(\omega) = \lim_{T\to\infty} \frac{E[X(-j\omega)Y(j\omega)]}{2T}$$

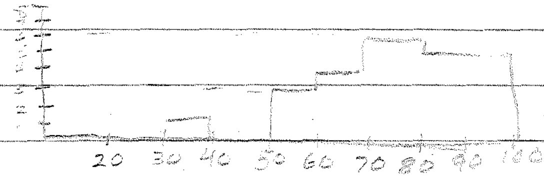
2-3-72

TEST TOMORROW CHAPT.'S 9-10-11

$\dfrac{1}{s+\alpha}$; $\dfrac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$; $\dfrac{\omega_0}{(s+\alpha)^2+\omega_0^2}$

2-8-71

TEST



AVER = 75.5

2-16-71

AUTOCORRELATION OF Y

$$R_y(\tau) = \int_0^\infty d\lambda_1 \int_0^\infty R_1(\lambda_2 - \lambda_1 - \tau)\, h(\lambda_1)\, h(\lambda_2)\, d\lambda_2$$

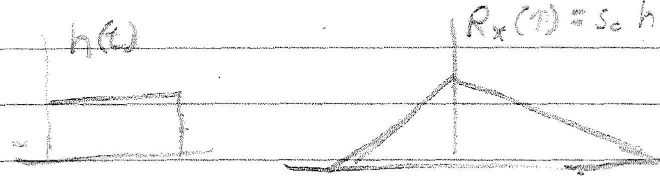
WHITE NOISE CASE: $R_x(\tau) = S_0\, \delta(\tau)$

$$R_y(\tau) = \int_0^\infty d\lambda_1 \int_0^\infty S_0\, \delta(\lambda_2 - \lambda_1 - \tau)\, h(\lambda_1)\, h(\lambda_2)\, d$$

$$= \int_0^\infty d\lambda_1\, S_0\, h(\lambda_1)\, h(\lambda_1 + \tau)$$

$$= S_0 \underbrace{\int_0^\infty h(\lambda_1)\, h(\lambda_1 + \tau)\, d\lambda_1}_{\text{CORRELATION FUNCTION}}$$

$$\underset{R_x(\tau) = S_0\delta(\tau)}{x(t)} \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$



CROSS-CORRELATION BETWEEN x & Y

$$R_{xy}(\tau) = E\left[x(t)\, y(t+\tau)\right]$$

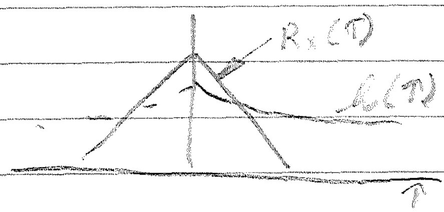
$$= E\left[x(t) \int_0^\infty x(t+\tau-\lambda)\, h(\ )\, d\lambda\right]$$

$$= E\left[\int_0^\infty x(t)\, x(t+\tau-\lambda)\, h(\ )\, d\lambda\right]$$

$$= \int_0^\infty \underbrace{E\left[x(t)\, x(t+\tau-\lambda)\right]}\, h(\lambda)\, d\lambda$$

$$= \int_0^\infty R_x(\tau-\lambda)\, h(\lambda)\, d\lambda \quad \text{- CONVOLUTION}$$

$$R_{yx}(\tau) = \int_0^\infty R_x(\tau+\lambda)\, h(\lambda)\, d\lambda \quad \text{- CORRELATION}$$

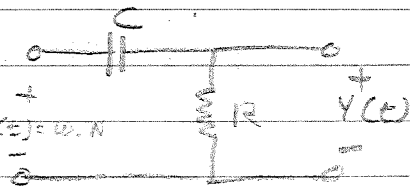


$$R_{xy}(\tau) = R_{xy}(-\tau)$$

FOR WHITE NOISE:

$$R_{XY}(\tau) = S_0 h(\tau) \; ; \; R_{YX}(\tau) = S_0 h(-\tau)$$ UP TO 12-6

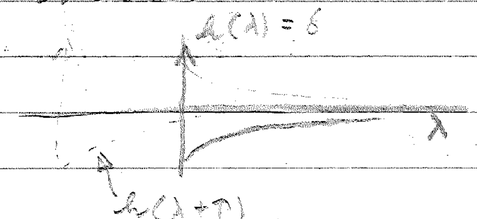
12-6)



$$R_Y(\tau) = \int_0^\infty d\lambda_1 \, S_0 h(\lambda_1) h_1(\lambda_1+\tau) d\lambda$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{R}{R + \frac{1}{sC}} = \frac{s}{s+b} = 1 - \frac{b}{s+b}$$

$$\therefore h(t) = \delta(t) - b e^{-bt} u(t)$$



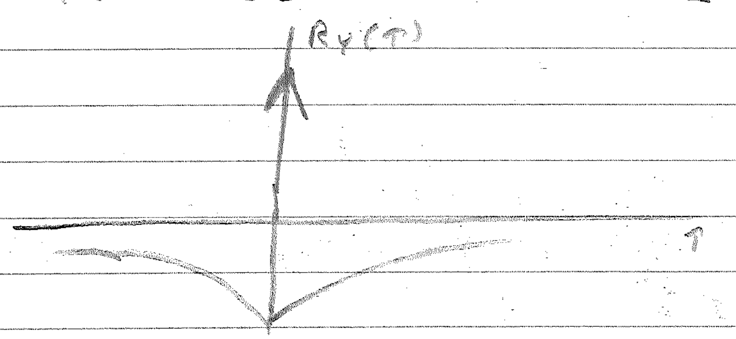
$$R_Y(\tau) = S_0 \int_0^\infty [\delta(\lambda) - b e^{-b\lambda}] \cdot [\delta(\lambda+\tau) - b e^{-b(\lambda+\tau)}] d\lambda$$

$$R_Y(\tau) = S_0 \int_0^\infty [\delta(\lambda)\delta(\lambda+\tau) + b\delta(\lambda) e^{-b(\lambda+\tau)} - b\delta(\lambda+\tau) e^{-b\lambda} + b^2 e^{-b\tau} e^{-2b\lambda}] d\lambda$$

$$= S_0 [\delta(\tau) - b e^{-b\tau} + 0 + \frac{b}{2} e^{-b\tau}] \quad \text{for } \tau \geq 0$$

$$= S_0 [\delta(\tau) - \frac{b}{2} e^{-b\tau}]$$

$$\Rightarrow R_Y(\tau) = S_0 [\delta(\tau) - \frac{b}{2} e^{-b|\tau|}]$$



$R_Y(\tau) = \int_0^\infty d\lambda_1 \int_0^\infty R_X(\lambda_2 - \lambda_1 - \tau)\, h(\lambda_2)\, h(\lambda_1)\, d\lambda_2$

$S_Y(\omega) = \mathcal{F}\{R_Y(\tau)\} = \int_{-\infty}^{\infty} \Big[\int_0^\infty d\lambda_1 \int_0^\infty R_X(\lambda_2 - \lambda_1 - \tau)\, h(\lambda_2)$
$h(\lambda_1)\, d\lambda_2\Big] e^{-j\omega\tau} d\tau$

$= \int_0^\infty d\lambda_1 \int_0^\infty d\lambda_2\, h(\lambda_2)\, h(\lambda_1) \int_{-\infty}^{\infty} R_X(\lambda_2 - \lambda_1 - \tau)\, e^{-j\omega\tau} d\tau$

$\mathcal{F}\{f(t-\alpha)\} = F(\omega)\, e^{-j\omega\alpha}$

$\Rightarrow S_Y(\omega) = S_X(\omega)\, e^{-j\omega(\lambda_2 - \lambda_1)}$

$= S_X(\omega) \int_0^\infty d\lambda_1 \int_0^\infty h(\lambda_2)\, h(\lambda_1)\, e^{-j\omega(\lambda_2 - \lambda_1)} d\lambda_2$

$= S_X(\omega) \int_0^\infty h(\lambda_1)\, e^{j\omega\lambda_1} d\lambda_1 \int_0^\infty h(\lambda_2)\, e^{-j\omega\lambda_2} d\lambda_2$

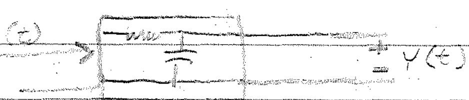
causal for $h(\lambda_1)$ & $\lambda_1 < 0 = 0$

$$\Rightarrow S_Y(\omega) = S_X(\omega)\, H(-\omega)\, H(\omega) = S_X(\omega)\, |H(j\omega)|^2$$

USING LAPLACE:

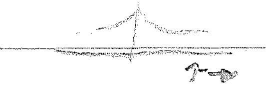
$S_Y(s) = S_X(s)\, H(s)\, H(-s)$

12-11) $X(t) = X_0 + n(t)$



$Y(t)$

$R_n(\tau) = e^{-|\tau|}$



WANT NOISE OUT R.M.S = .01 X & X = 1 V

$Y(t) = Y_1(\tau) + Y_2(t)$

FROM X_0 FROM n

$Y_1(\tau) = X_0$; $.01X = \sqrt{\overline{Y_2^2}} = 10^{-3}$

$\overline{Y^2} = \int_{\text{ALL } \omega} S_Y(\omega)$

$S_{Y_2} = H(s)\, H(-s)\, S_n(\omega)$

$H(s) = \dfrac{\frac{1}{sC}}{R + \frac{1}{sC}} = \dfrac{b}{s+b} \ni b = \dfrac{1}{RC}$

$\Rightarrow S_{Y_2} = \left(\dfrac{b}{s+b}\right)\left(\dfrac{b}{-s+b}\right)\left(\dfrac{-2}{s^2-1}\right) = \dfrac{2b^2}{(s+b)(s-b)(s+1)(s-1)}$

$\overline{Y_2^2} \Rightarrow I_2 = \dfrac{2b^2}{2b(b+1)1} = \dfrac{b}{b+1} = 10^{-6}$

$\therefore b \approx 10^{-6} = \dfrac{1}{RC}$

12-13) $H(s) = \dfrac{2s}{2s^2+7s+3}$; FIND $|H(j\omega)|^2$

$\Rightarrow H(s)\,H(-s) = \left(\dfrac{2s}{2s^2+7s+3}\right)\left(\dfrac{-2s}{2s^2-7s+3}\right)$

$= \dfrac{-4s^2}{4s^4-37s^2+9}$

$\Rightarrow |H(j\omega)|^2 = \dfrac{4\omega^2}{4\omega^4+37\omega^2+9}$

12-24) $S_n = \dfrac{10}{(\omega^2+1)(\omega^2+4)}$ $\xrightarrow{S_n(\omega)}$ WHITENING FILTER $\xrightarrow{S_y(\omega)}$ S_0

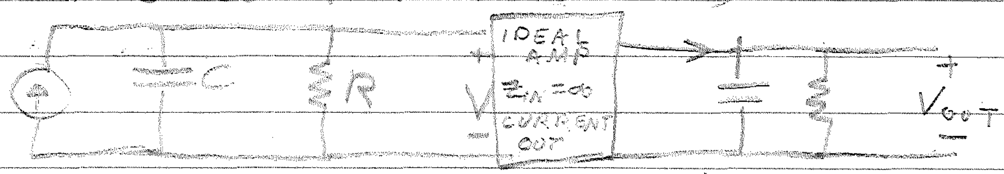
$S_y(s) = H(s)\,H(-s)\,S_x(s) = S_0 \Rightarrow H(s)H(-s) = \dfrac{S_0}{S_n(\omega)}$

$S_n(s) = \dfrac{10}{(s^2-1)(s^2-4)}$

$S_y(s) = \dfrac{S_0}{10}(s^2-1)(s^2-4)$

$= \left(\sqrt{S_0/10}\,(s+1)(s+2)\right)\left(\sqrt{S_0/10}\,(s-1)(s-2)\right)$

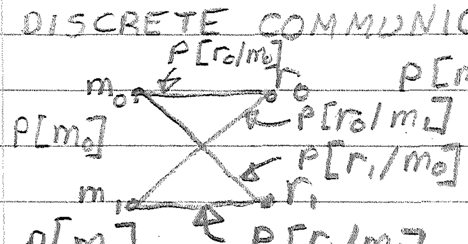
$\therefore H(s) = K(s^2+3s+2) = K(s+1)(s+2)$



THIS IS WRONG!

2-18-71

DISCRETE COMMUNICATION PROB:



$P[m_i]$ IS OPTION PROBLEM OF M_i

$P[m_i, r_j] = \underbrace{P[m_i/r_j]}_{\text{A POSTERIORI PROB OF } m_i} P[r_j] = P[r_j/m_i]\,P[m_i]$

LET THE DECISION OF WHAT WAS SENT BE $\hat{m}(j)$

CONDITIONAL PROBABILITY OF CORRECT DECISION $= P[c/r_j] = P[\hat{m}(j)/r_j]$

= PROBABILITY THAT CHOSEN SIGNAL ie $\hat{m}(j)$ WAS ACTUALLY TRANSMITTED

THUS, IF WE OPTIMIZE $P[c/r_j]$ BY CHOOSING $\hat{m}(j)$ FOR WHICH $P[m_j/r_j]$ IS BIGGEST

$P[c] = \sum_{\text{ALL } j} P[c/r_j]P[r_j]$

$\Rightarrow P[C]$ IS MAX IF ALL $P[c/r_j]$ ARE MAX.

DECISION: CHOOSE m_k IF $P[m_k/r_j] \geq P[m_i/r_j] \; \forall i \neq k$

OR $\dfrac{P[r_j/m_k]P[m_k]}{\cancel{P[r_j]}} \geq \dfrac{P[r_j/m_i]P[m_i]}{\cancel{P[r_j]}}$

OR $\underbrace{P[r_j/m_k]P[m_k] \geq P[r_j/m_i]P[m_i]}_{\text{BAYE'S RULE}}$

IF $P[m_i] = P[m_k] \; \forall i, k$, THEN

DECISION: $P[r_j/m_k] \geq P[r_j/m_i]$ CHOOSE M_k

(MAXIMUM LIKLEHOOD DECISION RULE)

FROM PREVIOS PROBLEM:

$P[m_0] = .6$; $P[m_1] = .4$; $P[r_0/m_0] = .2$ $P[r_0/m_1] = .7$

$P[r_1/m_0] = .8$ $P[r_1/m_1] = .3$

USING BAYE'S RULE:

$P[r_0/m_0]P[m_0] = .12$

$P[r_0/m_1]P[m_1] = .28$

$P[r_1/m_0]P[m_0] = .48$

$P[r_1/m_1]P[m_1] = .12$

GIVEN r_0 RECEIVED, CHOOSE $\hat{m}(j) = m_1$

" r_1 " " $\hat{m}(j) = m_0$

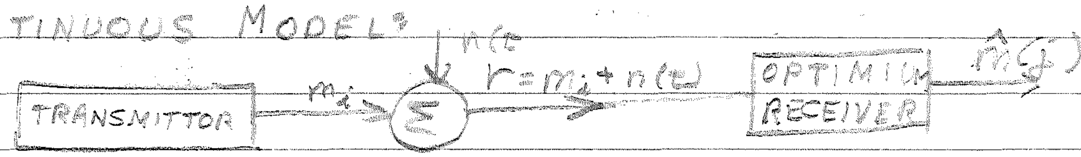
FIND CORRECTNESS

$P[c] = \sum_j P[c/r_j]P[r_j]$

$= .48 + .28 = .76$

$\Rightarrow P[\text{ERROR}] = .24$

CONTINUOUS MODEL:



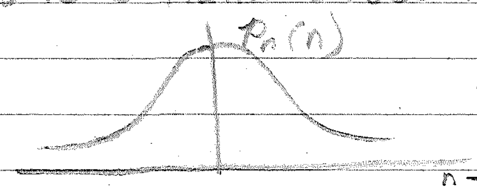
MAP r INTO THAT MESSAGE m_i FOR WHICH THE A POSTERIORI PROB IS MAX

OR $P[m_i/r] \geq P[m_j/r]$

OR $p_r[r/m_i]\,P[m_i] > p_r[r/m_j]\,P[m_j]$ (BAYES RULE)

$r = n + m_i \Rightarrow n = r - m_i$

ASSUME 0-MEAN GAUSSIAN NOISE

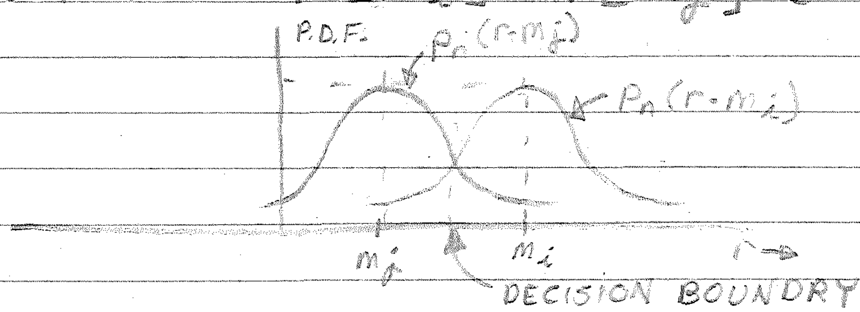


$\Rightarrow p_r(r/m_i) = p_n(r-m_i/m_i)$
$= p_n(r-m_i)$

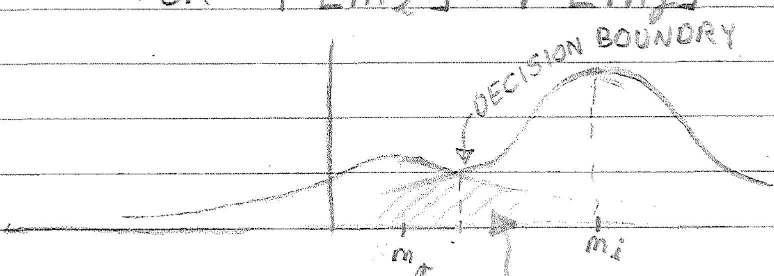
BY MODIFIED BAYES RULE:

$p_n(r-m_i)\,P[m_i] > p_n(r-m_j)\,P[m_j]$ (DECISION RULE)

SUPPOSE $P[m_i] = P[m_j] + c$



FOR $P[m_i] \neq P[m_j]$



ERROR = AREA UNDER CURVE

IF $P_n = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{n^2}{\sqrt{2\pi}\,\sigma}}$ USING BAYES MODIFIED:

$$\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(r-m_i)^2}{2\sigma^2}} P[m_i] > \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(r-m_j)^2}{2\sigma^2}} P[m_j]$$

TAKING $\ln$: $\frac{-(r-m_i)^2}{2\sigma^2} + \ln P[m_i] > \frac{-(r-m_j)^2}{2\sigma^2} + \ln P(m_j]$

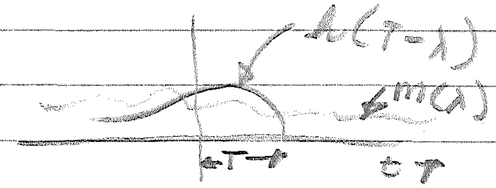
(CONT. 2-20-71)

2-22-71

FIND $h(t) \ni$ SIG TO NOISE RATIO IS MAX AT SOME TIME T

$$\xrightarrow{m(t)+n(t)} \boxed{h(t)} \xrightarrow{m_o(t)+n_o(t)}$$

$$m_o(T) = \int_{-\infty}^{T} m(\lambda) h(T-\lambda)\,d\lambda$$
$$n_o(T) = \int_{-\infty}^{T} n(\lambda) h(T-\lambda)\,d\lambda$$



$$R_o = \frac{m_o^2(T)}{E[n_o(T)]} = \overline{n_o^2(T)}$$

$$R_o = \frac{\left|\int_{-\infty}^{T} m(\lambda) h(T-\lambda)\,d\lambda\right|^2}{\int_{-\infty}^{T}\int_{-\infty}^{T} R_n(\lambda_2-\lambda_1) h(T-\lambda_2) h(T-\lambda_1)\,d\lambda_2\,d\lambda_1}$$

$R_n(\tau)$ MUST BE KNOWN. ONLY UNKNOWN IS $h(t)$

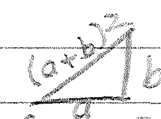
SUPPOSE $R_n(\tau) = \frac{N_o}{2}\delta(\tau) \Rightarrow S_n(\omega) = \frac{N_o}{2}$ (WHITE NOISE)

THEN $\int_{-\infty}^{T}\int_{-\infty}^{T} \frac{N_o}{2}\delta(\lambda_2-\lambda_1) h(T-\lambda_1) h(T-\lambda_2)\,d\lambda_2\,d\lambda_1$

$$= \int_{-\infty}^{T} \frac{N_o}{2} h(T-\lambda_2) h(T-\lambda_2)\,d\lambda_2$$
$$= \int_{-\infty}^{T} \frac{N_o}{2} h^2(T-\lambda_2)\,d\lambda_2 = \int_{-\infty}^{T} \frac{N_o}{2} h^2(T-\lambda)\,d\lambda$$

$$\therefore R_o = \frac{\left|\int_{-\infty}^{T} m(\lambda) h(T-\lambda)\,d\lambda\right|^2}{\frac{N_o}{2}\int_{-\infty}^{T} h^2(T-\lambda)\,d\lambda}$$

SHWARTZ'S INEQUALITY $\left|\int a(\lambda) b(\lambda)\,d\lambda\right|^2 \le \int a^2(\lambda)\,d\lambda \int b^2(\lambda)\,d\lambda$



$$\Rightarrow R_o \le \frac{\int_{-\infty}^{T} m^2(\lambda)\,d\lambda \cancel{\int_{-\infty}^{T} h^2(T-\lambda)\,d\lambda}}{\frac{N_o}{2} \cancel{\int_{-\infty}^{T} h^2(T-\lambda)\,d\lambda}}$$

$R_{o\,MAX}$ IS AT $m(\lambda) = h(T-\lambda)$

OR $m(T-\lambda) = h(\lambda)$ (MATCHED FILTER)

OR $H(j\omega) = M(-j\omega)\,e^{-j\omega T} \ni M(j\omega) = \mathcal{F}\{m(T)\}$

$$R_o = \frac{[m_o(T)]^2}{\overline{n_o^2(T)}}$$

$$\int_{-\infty}^{T} R_n(\lambda_2-\lambda_1)\, h(T-\lambda_1)\, d\lambda_1 = m(\lambda_2) \qquad -\infty \le \lambda_2 \le T$$

← WEIMER-HOPF INTEGRAL EQUATION

(GIVES OPT $h(t)$ FOR NON-WHITE NOISE)

LEAST MEAN-SQUARE ERROR RECEIVER

$$\left(\min:\ E\left[(m(t+\alpha)+m_o(t))^2\right]\right)$$

$$\int_0^{\infty} R_{rr}(\lambda_2-\lambda_1)\, h(\lambda_1)\, d\lambda_1 = R_{rm}(\lambda_2+\alpha)$$

$\ni$ R_{rr} = AUTOCORR. OF $r(t) = m(t) + n(t)$

& R_{rm} = CROSS CORREL. OF $r(t)$ & $m(t)$

FOR MATCHED FILTER & $h(t)$ NOT NECESSARILY REALIZABL

$$\int_{-\infty}^{\infty} R_n(\lambda_2-\lambda_1)\, h(T-\lambda_1)\, d\lambda_1 = m(\lambda_2)$$

$\mathcal{F}$ BOTH SIDES

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_n(\lambda_2-\lambda_1)\, h(T-\lambda_1)\, e^{-j\omega\lambda_2}\, d\lambda_2\, d\lambda_1 = \int_{-\infty}^{\infty} m(\lambda_2)\, e^{-j\omega\lambda_2}\, d\lambda_2$$

$$\Rightarrow S_n(\omega)\, H(-j\omega)\, e^{-j\omega T} = M(j\omega)$$

$$\Rightarrow H(-j\omega) = \frac{M(j\omega)\, e^{+j\omega T}}{S_n(j\omega)} \quad \text{OR} \quad H(j\omega) = \frac{M(-j\omega)\, e^{-j\omega T}}{S_n(j\omega)}$$

$$m(t)+n(t) \longrightarrow \boxed{\text{PRE-WHITENING FILTER}} \xrightarrow{m_1(t)\,+\,n_1(t)\ \ni\ S_{n_1} = K}$$

FOR P.W.F.; $H(s)$ = MATCH FILTER

$$\xrightarrow{m_1(t)+n_1(t)} \boxed{h(t)\ \text{MATCHED to}\ m_1(t)} \xrightarrow{y(t)}$$

$$\boxed{H(s)}\, H(s) = \frac{K}{S_x(s)} = \frac{K}{S_x^+(s)\, S_x^-(s)}$$

$$\Rightarrow H(s) = \sqrt{K} \,/\, S_x^+(s)$$

SUPPOSE $S_{n_1}(\omega) = 1$ $\quad$ $M_1(\omega) = H_1(\omega) M(\omega)$ $\quad$ LET H_2 BE MATCHED TO M_1

$S_n(s) H_1(s) H_1(-s) = 1$ $\quad$ ie $H_2(j\omega) = M_1(-j\omega) e^{-j\omega T}$

$\Rightarrow H_1(s) = \frac{1}{S_n^+(s)}$

$$\xrightarrow{m+n} \boxed{H_1(s) = \frac{1}{S_n(s)}} \xrightarrow[+n_1]{M_1} \boxed{H_2(\omega) = M_1(j\omega) e^{-j\omega T}} \longrightarrow$$

$\therefore H(\omega) =$ TRANSFORM OF WHOLE BLACK BOX

$$= H_1(\omega) H_2(\omega)$$

$$= \frac{1}{S_n^+(\omega)} M_1(-j\omega) e^{-j\omega T}$$

$$= \frac{1}{S_n(\omega)} \frac{M(j\omega) e^{-j\omega T}}{S_n(\omega)} = \frac{M(j\omega) e^{-j\omega T}}{S_n(j\omega)}$$

2-20-71

(2-19) $Y^2 = 31.42$

LECTURE:

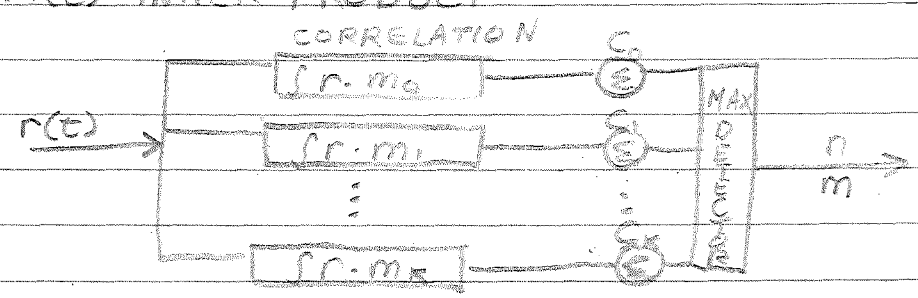
$$\frac{-r^2}{2\sigma^2} + \frac{2 m_i r}{\sigma^2} - \frac{m_i^2}{2\sigma^2} + \ln P[m_i] > \frac{-r^2}{2\sigma^2} + \frac{2 r m_j}{2\sigma^2} - \frac{m_j^2}{2\sigma^2} + \ln P[m_j]$$

$$r m_i + \underbrace{\left[\frac{-m_i^2}{2} + \sigma^2 \ln P[m_i]\right]}_{C_i} > r m_j + \underbrace{\left[\frac{-m_j^2}{2} + \sigma^2 \ln P[m_j]\right]}_{C_j}$$

$r \cdot m_i$ IS DOT PRODUCT FOR VECTOR VALUED FUNC.

EX) $r = [1, 1, -1, 1]$; $m_i = [1\,1\,1\,1]$; $m_j = [1\,1\,-1\,-1]$

$\int r(t)\, m(t)$ INNER PRODUCT

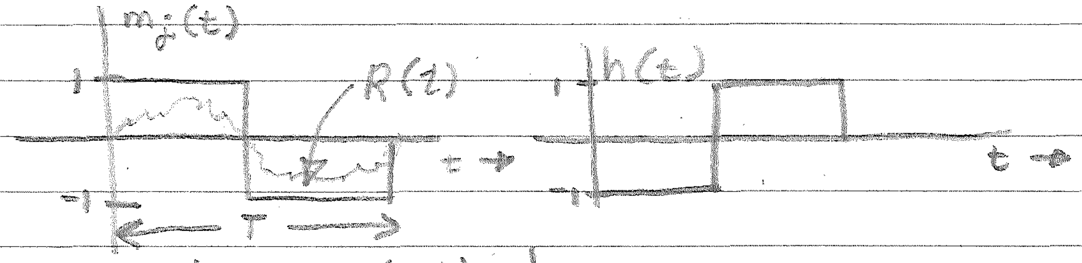


CORRELATION RECEIVER

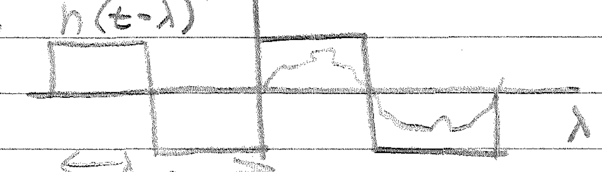
$x = r \rightarrow \boxed{h(t)} \rightarrow Y = \int r(t)\, m_i(t)\, dt = \int r(\lambda)\, m_i(\lambda)\, d\lambda$

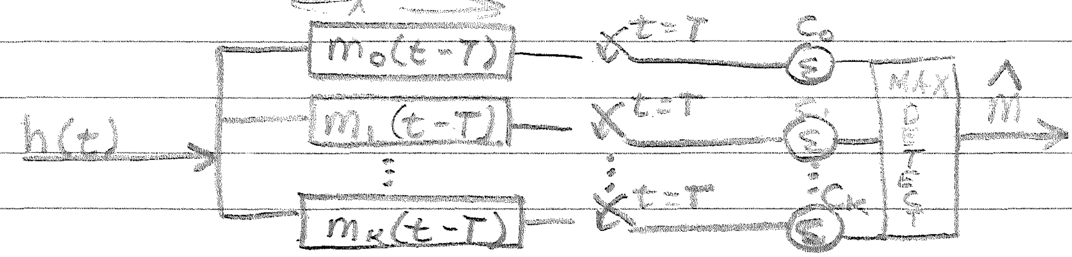
$Y(t) = \int x(\lambda)\, h(t-\lambda)\, d\lambda$

LET $h(t) = m_i(T+t) \Rightarrow m_i(t-\lambda) = h(t)$



$Y = r * h \Rightarrow$ $h(t-\lambda)$





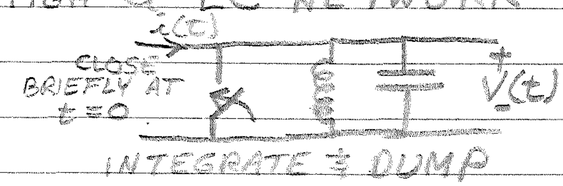
MATCHED FILTER RECEIVER

SUPPOSE $m_i(t) = \begin{cases} -\sqrt{\frac{2}{T}} \cos 2\pi f_i t & 0 \le t \le T \\ 0 & \text{ELSEWHERE} \end{cases}$

$\Rightarrow f_i = \frac{n}{T}$

$h_i(t) = m_i(T-t) = \begin{cases} \sqrt{\frac{2}{T}} \cos 2\pi f_i t & 0 \le t \le T \\ 0 & \text{ELSEWHERE} \end{cases}$

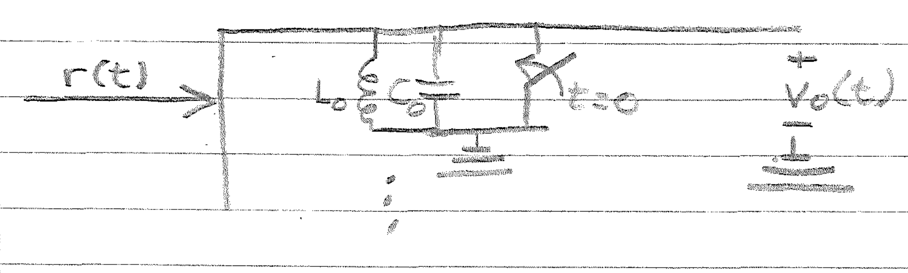
HIGH Q LC NETWORK



$L_i(t) = \frac{1}{C_i} \cos \frac{t}{\sqrt{L_i C_i}}$ $t \ge 0$

$C_i = \sqrt{\frac{T}{2}}$, $f_i = \frac{1}{2\pi\sqrt{L_i C_i}}$

INTEGRATE & DUMP



2-21-71

FINAL; 120 PTS

PROB ON STATE VARIABLE

CHAPTS 9-12 CHAPT 2 3 COMM. ~~&~~ TEST NO CROSS SPEC. DENSITY

PREWHITENING FILTER

MATCHED FILTER

APOSTEOURI DECISION RULE

CHECK HANDED IN PROB.

SOLUTION OF THE TRANSFORMED STATE EQUATION

If $\dot{x} = Ax + Bu$ then $X(s) = (sI - A)^{-1}\, x(0) + (sI - A)^{-1} B U(s)$

$$= \Phi(s)\, x(0) + G(s)\, U(s)$$

$$= X_{I.C.}(s) + X_F(s)$$

TIME DOMAIN SOLUTION OF THE STATE EQUATION

If $\dot{x} = Ax + Bu$ then $x(t) = \epsilon^{At}\, x(0) + \epsilon^{At} \int_0^t \epsilon^{-A\tau} Bu \, d\tau$

where $\epsilon^{At} = I + At + A^2 \frac{t^2}{2!} + \cdots + A^n \frac{t^n}{n!} + \cdots$

And, if y is the matrix of outputs,

$$y(t) = Cx + Du$$

ALGORITHM FOR COMPUTER SOLUTION

If $\dot{x} = Ax + Bu$ and if $u(t) = u(kT)$ for $kT \le t < \overline{k+1}\, T$

Then $x(\overline{k+1}\, T) = \phi(T)\, x(kT) + \Delta(T)\, u(kT)$

where $\phi(T) = \epsilon^{AT}$

$$= I + AT + A^2 \frac{T^2}{2!} + \cdots + A^n \frac{T^n}{n!} + \cdots$$

and $\Delta(T) = \epsilon^{AT} \int_0^T \epsilon^{-A\tau}\, d\tau\, B$

$$= \left[IT + A \frac{T^2}{2!} + A^2 \frac{T^3}{3!} + \cdots + A^n \frac{T^{n+1}}{(n+1)!} + \cdots \right] B$$

BOB MARKS

1/7/71 EE 344 94

Hour Test #1

RANGE 53-100

89 AVERAGE

Show all work on these problems

1) Put the following equation in state variable form:

$$\frac{d^3x}{dt^3} + K_2\frac{d^2x}{dt^2} + K_1 x = K f(t)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -K_1 & 0 & -K_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ K \end{bmatrix} u$$

$x_1 = x$

$x_2 = \dot{x}$

$x_3 = \ddot{x}$

$u = f(t)$

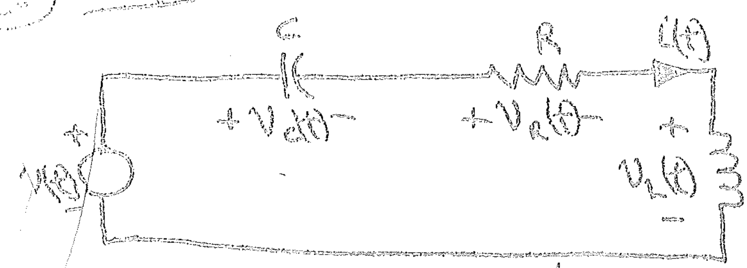
4/10

(II) 22 pts.

EE 374

$Y = \begin{bmatrix} V_R \\ V_L \end{bmatrix}$

p. 2



Here, $L\frac{di}{dt} + iR + \frac{1}{C}\int_0^t i(t)\,dt + V_c(0) = v(t)$

The desired outputs are $V_R = iR$ and $V_L = v(t) - \frac{1}{C}\int_0^t i(t)\,dt - iR - V_c(0)$

Defining the states and inputs as:

$x_1 = \int_0^t i(t)\,dt$, $x_2 = i(t)$, $u_1 = v(t)$, and $u_2 = V_c(0)$

Obtain the matrices $\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{C}$, and $\underset{\sim}{D}$ for $\dot{\underset{\sim}{x}} = \underset{\sim}{A}\,\underset{\sim}{x} + \underset{\sim}{B}\,\underset{\sim}{u}$ and $\underset{\sim}{y} = \underset{\sim}{C}\,\underset{\sim}{x} + \underset{\sim}{D}\,\underset{\sim}{u}$

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix}}_{A} X + \underbrace{\begin{bmatrix} 0 & 0 \\ \frac{1}{L} & -\frac{1}{L} \end{bmatrix}}_{B} U \begin{matrix} v_t \\ V_c \end{matrix}$$

$$\frac{di}{dt} = -\frac{R}{L} i - \frac{1}{LC}\int_0^t i(t)\,dt - \frac{V_c(0)}{L} + \frac{V(t)}{L}$$

$$\begin{bmatrix} V_R \\ V_L \end{bmatrix} = Y = \underbrace{\begin{bmatrix} 0 & R \\ -\frac{1}{C} & -R \end{bmatrix}}_{C} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}}_{D} U \begin{matrix} V(t) \\ V_c \end{matrix}$$

+20

(III) 16 pts

EE 374

Obtain a state and an output equation to represent the system described by

$$\frac{\Theta_F(s)}{R(s)} = \frac{K(s^2-2)}{s^4+4s^2-3s+6} = \frac{W}{R} \cdot \frac{\Theta}{W}$$

INPUT

$$\frac{W}{R} = \frac{K}{s^4+4s^2-3s+6} \Rightarrow Kr = \dddot{w} + 4\ddot{w} - 3\dot{w} + 6w$$

$$\dot{X} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -6 & 3 & -4 & 0 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ 0 \\ K \end{bmatrix} R$$

$x_1 = w$
$x_2 = \dot{w}$
$x_3 = \ddot{w}$
$x_4 = \dddot{w}$
$u_1 = r$

$$\frac{\Theta}{W} = s^2 - 2 \Rightarrow \Theta = \ddot{w} - 2w$$

$$Y = \begin{bmatrix} -2 & 0 & 1 & 0 \end{bmatrix} X$$

+16

(I) 24 pts

Given that $\underset{\sim}{X}(s) = [s\underset{\sim}{I} - \underset{\sim}{A}]^{-1}\underset{\sim}{x}(0) + [s\underset{\sim}{I} - \underset{\sim}{A}]^{-1}\underset{\sim}{B}\,\underset{\sim}{u}(s)$

and $\dot{\underset{\sim}{x}} = \overset{A}{\begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}} \underset{\sim}{x} + \overset{B}{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} \underset{\sim}{u}, \quad \underset{\sim}{x}(0) = \begin{bmatrix} 5 \\ -4 \end{bmatrix},$

and $\underset{\sim}{u} = u_1 = 3$; Find $\underset{\sim}{x}(t)$

$$sI - A \Rightarrow \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} s+2 & 0 \\ 0 & s+1 \end{bmatrix} \Rightarrow (s+1)(s+2)$$

$$\begin{bmatrix} s+1 & 0 \\ 0 & s+2 \end{bmatrix} \rightarrow \begin{bmatrix} s+1 & 0 \\ 0 & s+2 \end{bmatrix}$$

$$\Rightarrow (sI-A)^{-1} = \begin{bmatrix} \frac{1}{s+2} & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix}$$

$$X(s) = \begin{bmatrix} \frac{1}{s+2} & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} 5 \\ -4 \end{bmatrix} + \begin{bmatrix} \frac{1}{s+2} & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} \frac{3}{s} \\ -\frac{3}{s} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{s+2} \\ \frac{-4}{s+1} \end{bmatrix} + \begin{bmatrix} \frac{3}{s(s+2)} \\ \frac{-3}{s(s+1)} \end{bmatrix} = \begin{bmatrix} \frac{8}{s+2} \\ \frac{-7}{s+1} \end{bmatrix}$$

$$\Rightarrow X(t) = \cancel{\begin{bmatrix} 8e^{-2t} \\ -7e^{-t} \end{bmatrix}}$$

$$= \begin{pmatrix} 3.5\,e^{-2t} + 1.5 \\ -e^{-t} - 3 \end{pmatrix}$$

+20

EE 374 p-5

(V) 30 pts

Given the time domain solution for $\underline{x}(t)$ as

$$\underline{x}(t) = e^{\underline{A}t}\underline{x}(0) + e^{\underline{A}t}\int_0^t e^{-\underline{A}\tau}\underline{B}\,\underline{u}(\tau)\,d\tau$$

Find $\underline{x}(0.2)$, given $\dot{\underline{x}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A}\underline{x} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B}\underline{u}$,

with $\underline{x}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\underline{u} = u_1 = 500t$

Use only the first two terms of the series expansion for $e^{\underline{A}t}$.

$$e^{A.2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & .2 \\ -.2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & .2 \\ -.2 & 1 \end{bmatrix} \qquad e^{A.2}X(0) = \begin{bmatrix} 2.2 \\ .6 \end{bmatrix}$$

$$e^{-A\tau} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\tau \\ +\tau & 0 \end{bmatrix} = \begin{bmatrix} 1 & -\tau \\ +\tau & 1 \end{bmatrix}$$

$$e^{-A\tau}BU(\tau) = \begin{bmatrix} 1 & -\tau \\ +\tau & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 500\tau \end{bmatrix} = \begin{bmatrix} -500\tau^2 \\ 500\tau \end{bmatrix}$$

$$\int_0^t = \begin{bmatrix} \frac{-500t^3}{3} \\ 250t^2 \end{bmatrix}_{.2} = \begin{bmatrix} -\frac{4}{3} \\ 10 \end{bmatrix}$$

$$e^{.2A}\int_0^t = \begin{bmatrix} 1 & .2 \\ -.2 & 1 \end{bmatrix}\begin{bmatrix} -\frac{4}{3} \\ 10 \end{bmatrix} = \begin{bmatrix} 2/3 \\ \cancel{38/3} \\ 10.267 \end{bmatrix}$$

$$X(.2) = \begin{bmatrix} 2.2 \\ .6 \end{bmatrix} + \begin{bmatrix} 2/3 \; \cancel{3.7} \\ 10.267 \end{bmatrix} = \cancel{\begin{bmatrix} 5.9 \\ 13.3 \end{bmatrix}} = \begin{pmatrix} 2.867 \\ 10.867 \end{pmatrix}$$

500 .008 4000 750 .04 10,000

+28

BOB MARKS

79

Hr. Test #2

SHOW ALL WORK   25 pts per page

(1.) A random phase angle Θ has a uniform probability density function given by

$$p(\theta) = \begin{cases} \frac{1}{2\pi} & 0 \le \theta \le 2\pi \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Find $E[\theta] = \overline{\theta}$

$$E(\theta) = \int_{-\infty}^{\infty} \theta p(\theta) d\theta$$

$$E(\theta) = \int_0^{2\pi} \frac{\theta}{2\pi} d\theta = \left. \frac{\theta^2}{4\pi} \right]_0^{2\pi} = \frac{4\pi^2}{4\pi} = \pi$$

(b) Find $E[\theta^2] = \overline{\theta^2}$

$$E(\theta^2) = \int_{-\infty}^{\infty} \theta^2 p(\theta) d\theta$$

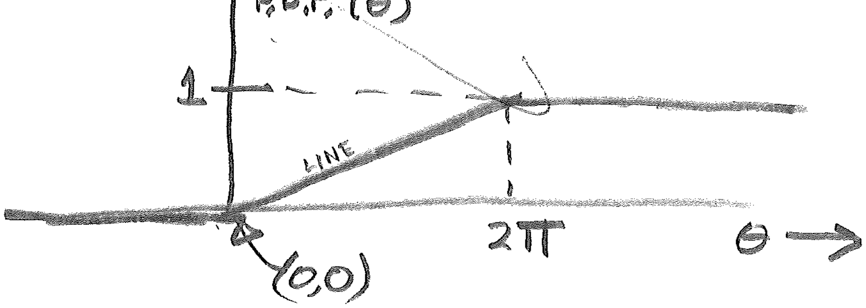
$$= \int_0^{2\pi} \frac{\theta^2}{2\pi} d\theta = \left. \frac{\theta^3}{6\pi} \right]_0^{2\pi} = \frac{8\pi^3}{6\pi} = \frac{4}{3}\pi^2$$

(c) Find σ_θ^2

$$\sigma_\theta^2 = \overline{X^2} - \overline{X}^2 = \frac{4}{3}\pi^2 - \pi^2 = \frac{1}{3}\pi^2$$

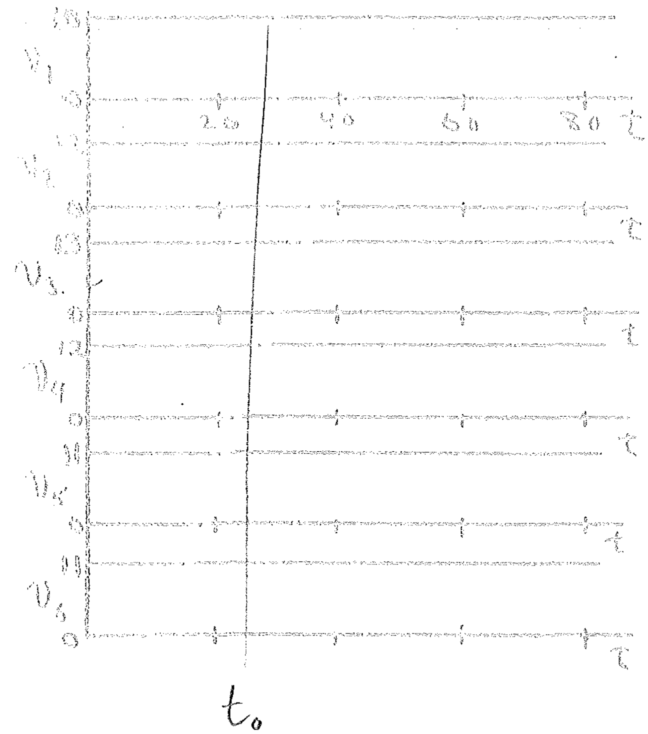
(d) Sketch the probability distribution function of θ. Label break points.

$$\text{P.D.F.}(\theta) = \int_{-\infty}^{\infty} p(\theta) d\theta$$



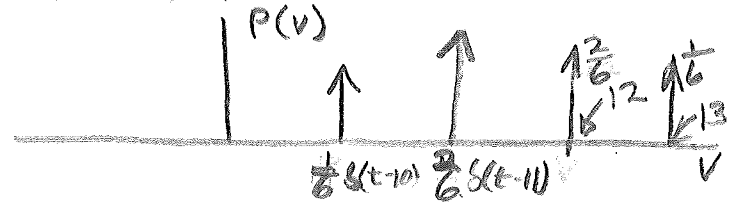
+25

II.) Suppose I take voltage vs. time measurements on a number of nominally 12 volt batteries and get the following curves.



(a) What is the ensemble average of $V(20) = E[V(20)]$?

10, 12, 13, 12, 11, 11



$\Rightarrow E[V(20)] = 6\overline{)10+12+13+12+11+11}$

$= \frac{69}{6} = \underline{11.5 \text{ volts}}$

(b) What is the time average of $V_3(t) = \langle V_3(t) \rangle$?

$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} V_3(t)\,dt$

$\lim \frac{1}{2T}\int_{-T}^{T} 13\,dt$

$\lim_{T\to\infty} \frac{1}{2T}\, 26T = 13$

(c) Is this process stationary? Give reason for your answer.

YES. DOESN'T MATTER THE POSITION OF t_0 (SAMPLING TIME), FOR THE DATA OBTAINED WILL BE THE SAME

(d) Is this process ergodic? Give reason for your answer.

~~A SINGLE MEMBER OF THE ENSEMBLE WOULD BE~~

NO. EVERY MEMBER OF THE ENSEMBLE DOES NOT EXHIBIT THE SAME AVERAGE VALUE ACROSS A MEMBER AS IT DOES DOWN THE ENSEMBLE ($R_x(\tau) \neq \mathcal{R}_x(\tau)$)

+20

(11) Given $R_X(\tau) = 2e^{-3|\tau|} + .44\cos\omega_0\tau + 1$

(a) Find $\bar{X}$ D.C. VALUE $R_X(\tau) = 1 \Rightarrow \bar{X}^2 = \sqrt{1} = 1 \Rightarrow \bar{X} = 1$

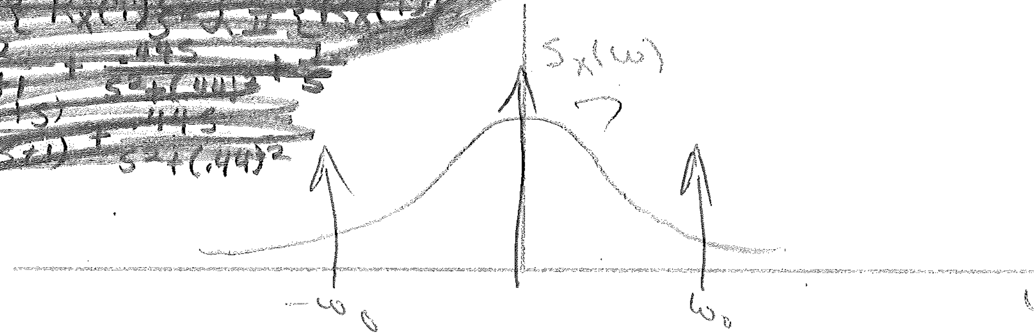
(b) Find $\overline{X^2} = R_X(0) = 2 + .44 + 1 = 3.44$

$$S_X(s) = \frac{2}{s+3} + \frac{.44s}{s^2+\omega_0^2} + \frac{1}{s}$$

$$S_X(s) = \frac{(3s+3)(s^2+\omega_0^2) + .44s^2(s+3)}{s(s+3)(s^2+\omega_0^2)}$$

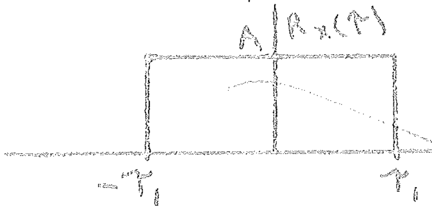
(c) Sketch $S_X(\omega)$ vs. ω. Don't worry about scaling the vertical axis.

$\frac{(s+a)^2}{(s+a)^2+\omega_0^2}$ cos

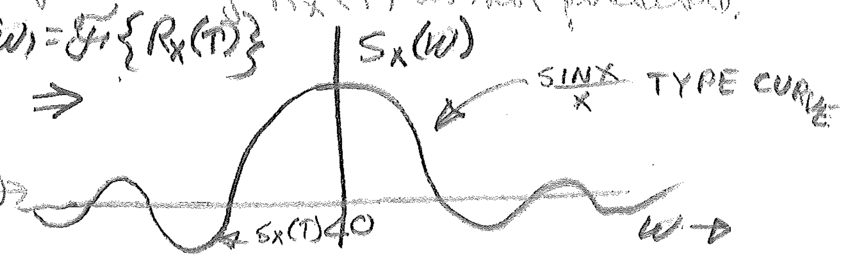


$\frac{s}{s^2+a^2} \to \cos$ $\quad \frac{a}{s^2+a^2}$

(d) Using the properties of correlation functions and spectral densities, show why the following $R_X(\tau)$ is not possible.



$S_X(\omega) = \mathcal{F}\{R_X(\tau)\} \Rightarrow$



$S_X(\tau) \geq 0 \Rightarrow$ FUNCTION NO GOOD

.44
.44
176
176
.1936

+19

P-4

Given $S_x(\omega) = \frac{2}{\omega^2+4}$ [redacted]

(a) Find $S_x(s) \ni \frac{2}{-s^2+4}$

(b) Find $\overline{X^2}$ from $S_x(s)$ using either contour integration or the table 11-1 formulas.

This would be $\overline{X^2}$, residue of LHP pole.

$$S_x(s) = \frac{2}{-s^2+4} = \frac{2}{(-s+2)(s+2)} = \frac{A}{+s+2} + \frac{\overline{X^2}}{-s+2}$$

$$\cancel{2 = A(-s+2) + \overline{X^2}(+s+2)}$$

$$\cancel{s=2 \Rightarrow 2 = 4\overline{X^2} \Rightarrow \overline{X^2} = \frac{1}{2}}$$

$$S_x(s) = \frac{2}{(2+s)(2-s)} = \left(\frac{\sqrt{2}}{2+s}\right)\left(\frac{\sqrt{2}}{2-s}\right)$$

(c) Find $R_x(\tau)$ from $S_x(s)$

$$S_x(s) = \frac{2}{-s^2+4}$$

$$s \to -s$$

$$S_x(s) = \frac{2}{-s^2+4} = \frac{-\frac{1}{2}}{(s-2)} + \frac{\frac{1}{2}}{(s+2)}$$

$$\Rightarrow R_x(-\tau) = +\frac{1}{2}e^{2t} + \frac{1}{2}e^{-2t}$$

$$R_x(\tau) = +\frac{1}{2}e^{-2\tau} + \frac{1}{2}e^{+2\tau}$$

$\tau \leq 0$ $\tau \geq 0$

$$R_x(\tau) = \frac{1}{2}e^{-2|\tau|}$$ ← impossible.

+15

Homework # 2 Solution

8.6 $\quad X = A\cos(\omega t + \theta)\,, \quad y = B\sin(\omega t + \theta)$

a) where $p(\theta) = \begin{cases} \frac{1}{2\pi} & 0 \le \theta \le 2\pi \\ 0 & \text{elsewhere} \end{cases}$

$$R_{xy}(\tau) = E\left[AB\cos(\omega t + \theta)\sin(\omega t + \omega\tau + \theta)\right]$$

$$= \int_0^{2\pi} \frac{AB}{2\pi}\cos(\omega t + \theta)\sin(\omega t + \omega\tau + \theta)\,d\theta$$

$$= \int_0^{2\pi} \frac{AB}{4\pi}\left[\sin(2\omega t + \omega\tau + 2\theta) + \sin(\omega\tau)\right]d\theta$$

$$= \underline{\frac{AB}{2}\sin(\omega\tau)}$$

$$R_{yx}(\tau) = E\left[AB\sin(\omega t + \theta)\cos(\omega t + \omega\tau + \theta)\right]$$

$$= \int_0^{2\pi} \frac{AB}{4\pi}\left[\sin(2\omega t + \omega\tau + 2\theta) + \sin(-\omega\tau)\right]d\theta$$

$$= -\frac{AB}{2}\sin(\omega\tau) = R_{xy}(-\tau)$$

b) $R_{yx}(0) = R_{xy}(0) = 0 \Rightarrow$ X & y are uncorrelated at $\tau = 0$ (orthogonal)

2.8) $p(x) = Ae^{-bx}, \quad x \geq 0$

a) $\int_0^\infty Ae^{-bx}\,dx = 1 = -\frac{A}{b}e^{-bx}\Big|_0^\infty = \frac{A}{b} \Rightarrow A = b$

b) $\text{Prob}(0 \leq x \leq 1) = \int_0^1 e^{-x}\,dx = 1 - e^{-1} = .632$

c) $P(x) = \int_0^x e^{-u}\,du = -e^{-u}\Big|_0^x = 1 - e^{-x}$

2.9) a) $p(x) = .1e^{-.1x}$

$$\underline{\bar{x}} = \int_0^\infty x(.1)e^{-.1x}\,dx = -x\cancel{e^{-.1x}}\Big|_0^\infty + \int_0^\infty e^{-.1x}\,dx = \underline{10}$$

b) $$\underline{\overline{x^2}} = \int_0^\infty x^2(.1)e^{-.1x}\,dx = -x^2\cancel{e^{-.1x}}\Big|_0^\infty + 2\int_0^\infty xe^{-.1x}\,dx$$

$= \underline{200}$ using integration result of (a).

c) $$\underline{\sigma_x^2} = E[x^2] - \{E[x]\}^2$$

$$= \overline{x^2} - (\bar{x})^2 = 200 - 100 = \underline{100}$$

11-3 In general: $S_x(\omega) = |F(j\omega)|^2 \left[\frac{\sigma_a^2}{t_1} + \frac{2\pi(\bar{a})^2}{t_1^2}\sum_{n=-\infty}^{\infty}\delta(\omega \ldots\right.$

From m, $p(a) = \frac{1}{4} a e^{-\frac{a^2}{8}} u(a)$

Then $\bar{a} = \int_0^\infty \frac{1}{4} a^2 e^{-\frac{a^2}{8}} da = \frac{1}{4}\,\frac{\sqrt{\pi}}{4}\cdot 8^{\frac{3}{2}} = \sqrt{2\pi}$

$\overline{a^2} = \int_0^\infty \frac{1}{4} a^3 e^{-\frac{a^2}{8}} da = 8$

$\sigma_a^2 = \overline{a^2} - (\bar{a})^2 = 8 - (\sqrt{2\pi})^2 = 8 - 2\pi$

$\therefore S_x(j\omega) = |F(j\omega)|^2 \left[\frac{8-2\pi}{t_1} + \frac{(2\pi)^2}{t_1^2}\sum_{n=-\infty}^{\infty}\delta\left(\omega - \frac{2\pi n}{t_1}\right)\right]$

If the pulses are rectangular with width t_1 & amplitude 1:

$F(j\omega) = \frac{t_1}{8}\,\frac{\sin\frac{\omega t_1}{16}}{\omega\frac{t_1}{16}}$

and finally

$S_x(j\omega) = \left(\frac{t_1}{8}\right)^2\left(\frac{\sin\frac{\omega t_1}{16}}{\frac{\omega t_1}{16}}\right)^2\left[\frac{8-2\pi}{t_1} + \frac{(2\pi)^2}{t_1^2}\sum_{n=-\infty}^{\infty}\delta\left(\omega - \frac{2\pi n}{t_1}\right)\right]$

$$X(t) = A\cos(\omega t + \theta) \quad \text{where} \quad p(\theta) = \begin{cases} \frac{1}{2\pi} & -\pi \le \theta \le \pi \\ 0 & \text{elsewhere} \end{cases}$$

$$R_x(\tau) = E[X(t)\,X(t+\tau)] = \int_{-\pi}^{\pi} \frac{A^2}{2\pi}\cos(\omega t + \theta)\cos(\omega t + \tau + \theta)\,d\theta$$

$$= \frac{A^2}{4\pi}\int_{-\pi}^{\pi}\left[\cos\tau + \cos(2\omega t + 2\theta + \tau)\right]d\theta$$

$$= \frac{A^2}{4\pi}\,\theta\cos\omega\tau\Big|_{-\pi}^{\pi} + 0 = \frac{A^2}{2}\cos\omega\tau$$

$$\overline{y^2} = \int_0^\infty d\lambda_1 \int_0^\infty R_x(\lambda_2 - \lambda_1)\,h(\lambda_1)\,h(\lambda_2)\,d\lambda_2 \quad \text{and } h(t) = u(t) - u(t-T)$$

$$\overline{y^2} = \int_0^T d\lambda_1 \int_0^T \frac{A^2}{2}\cos\omega(\lambda_2 - \lambda_1)\,d\lambda_2$$

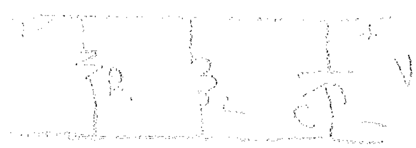
$$= \int_0^T d\lambda_1 \left\{\frac{A^2}{2\omega}\sin\omega(\lambda_2 - \lambda_1)\Big|_0^T\right\}$$

$$= \int_0^T \frac{A^2}{2\omega}\left\{\left[\sin(\omega(T - \lambda_1))\right] + \sin\omega\lambda_1\right\}d\lambda_1$$

$$= \frac{A^2}{2\omega^2}\left[\cos\omega(T - \lambda_1)\Big|_0^T - \cos\omega\lambda_1\Big|_0^T\right]$$

$$= \frac{A^2}{\omega^2}\left[1 - \cos\omega T\right]$$

$$\overline{y^2} = \frac{2A^2}{\omega^2}\sin^2\left(\frac{\omega T}{2}\right)$$



$$H(s) = \frac{V(s)}{I(s)} = \frac{1}{Y(s)} = \frac{1}{\frac{1}{R} + s\text{[illegible]}}$$

[illegible] $= 10^{-6}$ $\frac{\text{[illegible]}}{\text{cycle}}$

$$= \frac{1}{C}\,\frac{s}{s^2 + \frac{s}{RC} + \frac{1}{LC}}$$

[illegible] Bandwidth $= \frac{\omega_0}{Q} = \frac{\omega_0}{\omega_0 RC} = \frac{1}{RC}$

[illegible] $H(s) = \frac{\frac{1}{C}s}{s^2 + Bs + \omega_0^2} = \frac{Ks}{s^2 + Bs + \omega_0^2}$ where $K = \frac{1}{C}$

[illegible] $= 20 \log A_{mid} \Rightarrow A_{mid} = \sqrt{10^3} = \sqrt{1000}$

[illegible] $A_{mid} = H(j\omega_0) = \frac{Kj\omega_0}{-\omega_0^2 + jB\omega_0 + \omega_0^2} = \frac{K}{B} = \sqrt{1000}$

[illegible] $= H(s)\,H(-s)\,S_I(s) = \left(\frac{Ks}{s^2 + Bs + \omega_0^2}\right)\frac{(-Ks)}{(s^2 - Bs + \omega_0^2)} \times 10^{-6}$

$\overline{V^2} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} S_V(s)\,ds$, use I_2 where $c_0 = 0$, $c_1 = K \times 10^{-3}$

$d_0 = \omega_0^2$, $d_1 = B$, $d_2 = 1$

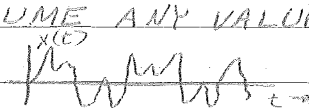
$\overline{V^2} = I_2 = \frac{c_1^2 d_0 + c_0^2 d_2}{2 d_0 d_1 d_2} = \frac{K^2 \times 10^{-6} \times \omega_0^2}{2(\omega_0^2)(B)(1)} = \frac{K^2}{2B} \times 10^{-6} = \frac{B}{2}(\text{[illegible]}$

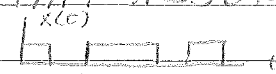
$= \frac{1}{2} \times (2\pi \times 10^4) \times (\sqrt{1000})^2 \times 10^{-6} = 10\pi$

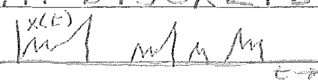
$\therefore V_{rms} = \sqrt{31.4}$ volts noise output of this tuned ci[illegible]

I) RANDOM SIGNALS AND PROBABILITY

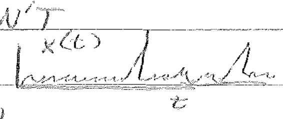
A) CONTINUOUS VS DISCRETE RANDOM PROCESS

1) CONTINUOUS - FUNCTION MAY ASSUME ANY VALUE WITHIN A SPECIFIED RANGE 

2) DISCRETE - RANDOM VARIABLE MAY ASSUME ONLY SPECIFIC VALUES 

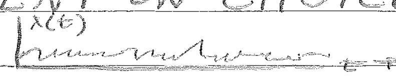
3) MIXED - HAVE BOTH DISCRETE & CONTINUOUS COMPONENTS 

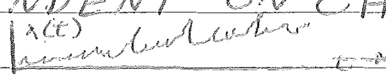
B) DETERMINISTIC VS NONDETERMINISTIC RANDOM PROCESSES

1) NONDETERMINISTIC - FUTURE VALUES CAN'T BE PREDICTED KNOWING PAST VALUES 

2) DETERMINISTIC - FUTURE VALUES CAN BE PREDICTED KNOWING ALL PAST VALUES

C) STATIONARY VS NONSTATIONARY RANDOM PROCESSES

1) STATIONARY - MARGINAL & JOINT DENSITY FUNCTIONS NOT DEPENDENT ON CHOICE OF THE TIME ORIGIN 

2) NONSTATIONARY - DEPENDENT ON CHOICE OF THE TIME ORIGIN 

D) ERGODIC VS NONERGODIC RANDOM PROCESS

1) ERGODIC - STATIONARY PROCESS IN WHICH EVERY MEMBER OF ENSEMBLE EXHIBITS SAME STATISTICAL BEHAVIOR AS WHOLE

a) MEAN VALUES & MOMENTS MAY BE COMPUTED BY TIME AVERAGES, AS WELL AS ENSEMBLE AVERAGES

b) $\overline{X^n} = \int_{-\infty}^{\infty} x^n p(x)\,dx = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x^n(t)\,dt$

2) NON-ERGODIC - NOT ERGODIC

II) CORRELATION FUNCTIONS

A) INTRODUCTION

1) CORRELATION FUNCTION - THE EXPECTED VALUE OF TWO RANDOM VARIABLES

2) AUTOCORRELATION FUNCTION - TWO RANDOM VARIABLES FROM SAME RANDOM PROCESS

a) $R_x(t_1,t_2) = E[x_1 x_2] = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} x_1 x_2 \, p(x_1,x_2)\, dx_2$

b) FOR A STATIONARY PROCESS

① $R_x(t_1,t_2) = R_x(t_1+T,\, t_2+T)$
$= E[x(t_1+T)\, x(t_2+T)]$

② SETTING $T = -t_1$
$R_x(t_1,t_2) = E[x(0)\, x(t_2-t_1)]$

③ SETTING $t_2 - t_1 = \tau$
$R_x(\tau) = E[x(t)\, x(t+\tau)]$

3) TIME-AUTOCORRELATION FUNCTION

a) $\mathcal{R}_x(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, x(t+\tau)\, dt$
$= \langle x(t)\, x(t+\tau) \rangle$ (AVERAGE

b) FOR ERGODIC PROCESS:
$\mathcal{R}_x(\tau) = R_x(\tau)$

4) CORRELATION COEFFICIENT (ρ)

a) DERIVATION FROM ZERO-MEAN STATIONARY PROCESS:

$Y(t) = x(t) - \rho x(t+\tau)$

$E[Y^2(t)] = E[x^2(t) - 2\rho x(t)\, x(t+\tau) + \rho^2 x^2(t+\tau)]$

$\Rightarrow \sigma_Y^2 = \sigma_x^2 - 2\rho R_x(\tau) + \rho^2 \sigma_x^2$

$\frac{d\sigma_Y^2}{d\rho} = -2R_x(\tau) + 2\rho\sigma_x^2 = 0$

$\Rightarrow \rho = R_x(\tau) / \sigma_x^2$

b) ρ MEASURES CORRELATION BETWEEN TWO RANDOM VARIABLES ($-1 \le \rho \le 1$)

① $\rho = 1 \Rightarrow$ IDENTICAL CORRELATION

② $\rho = 0 \Rightarrow$ COMPLETELY UNCORRELATED

③ $\rho = -1 \Rightarrow$ IDENTICAL WITH OPPOSITE SIDES

B) PROPERTIES OF AUTOCORRELATION FUNCTIONS

1) $R_X(0) = \overline{X^2}$

2) $R_X(\tau) = R_X(-\tau) \Rightarrow R_X(\tau)$ IS AN EVEN FUNCTION

3) $R_X(0) \geq |R_X(\tau)|$

4) $X(t)$ CONTAINS D.C. COMPONENT $\Rightarrow R_X(\tau)$ HAS D.C. COMP.

a) $X(t) = X_0 \Rightarrow R_X(\tau) = X_0^2$

b) $X(t)$ CONTAINS D.C. COMPONENT & ZERO-MEAN COMP

$\Rightarrow R_X(\tau) = X_0^2 + R_n(\tau)$

5) $X(t)$ CONTAINS PERIODIC COMPONENT

$\Rightarrow R_X(\tau)$ CONTAINS PERIODIC COMPONENT

6) IF $X(t)$ HAS NO PERIODIC COMPONENT, THEN

$\lim_{|\tau| \to \infty} R_X(\tau) = 0$

C) MEASUREMENT OF AUTOCORRELATION FUNCTIONS

1) ESTIMATED CORRELATION FUNCTION

$\hat{R}_X(\tau) = \frac{1}{T-\tau} \int_0^{T-\tau} X(t)\, X(t+\tau)\, dt \ni \tau << T$

2) COMPUTOR METHOD

$\hat{R}(n\Delta t) = \frac{1}{N-n} \sum_{k=0}^{N-n} X_k X_{k+n} \ni n = 0, 1, 2 \ldots M$ & $M << N$

D) CROSS CORRELATION FUNCTIONS

1) $R_{XY}(\tau) = E[X_1 Y_2] = \int_{-\infty}^{\infty} dX_1 \int_{-\infty}^{\infty} X_1 Y_2\, p(X_1 Y_2)\, dY_2$

$\ni X_1$ & Y_2 ARE JOINTLY STATIONARY

2) TIME CROSS-CORRELATION FUNCTION

$\mathcal{R}_{XY}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, y(t+\tau)\, dt$

3) FOR JOINTLY ERGODIC PROCESSES

$\mathcal{R}_{XY}(\tau) = R_{XY}(\tau)$

E) PROPERTIES OF CROSS-CORRELATION FUNCTIONS

1) $R_{XY}(0) = R_{YX}(0)$, BUT NEITHER HAS ANY PARTICULAR PHYSICAL VALUE

2) NO PARTICULAR SYMMETRY, BUT $R_{XY}(\tau) = R_{YX}(-\tau)$

3) $|R_{XY}(\tau)| \leq [R_X(0)\, R_Y(0)]^{1/2}$

4) FOR STATISTICALLY INDEPENDENT X & Y; $R_{XY}(\tau) = R_{YX}(\tau)$

EITHER X & Y ZERO-MEAN $\Rightarrow R_{XY}(\tau) = 0$

F) SUMS OF RANDOM PROCESSES

$$R_Z(\tau) = R_{X\pm Y}(\tau) = R_X(\tau) + R_Y(\tau) \pm R_{XY}(\tau) \pm R_{YX}(\tau)$$

G) CORRELATION FUNCTIONS FOR RANDOM VECTORS

1) FOR VECTOR $X(t)$, $R_X(t_1, t_2) = E[X(t_1) X^T(t_2)]$

$$R_X(t_1,t_2) = E\begin{bmatrix} X_1(t_1)X_1(t_2) & X_1(t_1)X_2(t_2) & \cdots & X_1(t_1)X_m(t_2) \\ X_2(t_1)X_1(t_2) & X_2(t_1)X_2(t_2) & & \\ \vdots & & & \\ X_m(t_1)X_1(t_2) & \cdots & \cdots & X_m(t_1)X_m(t_2) \end{bmatrix}$$

$$= \begin{bmatrix} R_{11}(t_1t_2) & R_{12}(t_1,t_2) & \cdots & R_{1m}(t_1t_2) \\ R_{21}(t_1t_2) & \cdots & \cdots & \vdots \\ \vdots & & & \\ R_{m1}(t_1t_2) & \cdots & \cdots & R_{mm}(t_1t_2) \end{bmatrix}$$

~~2) PROPERTIES OF $R_X(t_1t_2)$ MATRIX~~

~~a) ALL $R_{ii}(t_1)$~~

2) FOR STATIONARY PROCESS $(t_2 = t_1 + \tau)$

$$R_X(\tau) = \begin{bmatrix} R_{11}(\tau) & R_{12}(\tau) & \cdots & R_{1m}(\tau) \\ R_{21}(\tau) & & & \vdots \\ \vdots & & & \\ R_{m1}(\tau) & \cdots & \cdots & R_{mm}(\tau) \end{bmatrix}$$

a) ALL $R_{ii}(\tau)$ ARE EVEN FUNCTIONS

b) $R_{ij}(\tau) = R_{ji}(-\tau)$; $R_{ij}(0) = R_{ji}(0)$

c) IF ALL COMPONENTS ARE ZERO-MEAN & STATISTICALLY INDEPENDENT $R_{ij}(\tau) = 0$

3) FOR AUTOCORRELATION FUNCTIONS

$$R_{XY}(t_1,t_2) = E[X(t_1)Y^T(t_2)]$$

$$R_{XY}^{(t_1t_2)} = \begin{bmatrix} R_{X_1Y_1}(t_1t_2) & \cdots & R_{X_1Y_p}(t_1t_2) \\ \vdots & & \\ R_{X_mY_1}(t_1t_2) & \cdots & R_{X_mY_p}(t_1t_2) \end{bmatrix}$$

H) CORRELATION MATRICES FOR SAMPLED FUNCTION

A) FUNTION SAMPLED PERIODICALLY & IS SET IN VECTOR NOTATION.

B) R_1

$$R_x = E[XX^T] = E\begin{bmatrix} x(t_1)x(t_1) \cdots x(t_1)x(t_n) \\ x(t_2)x(t_1) \quad \vdots \\ x(t_n)x(t_1) \cdots x(t_n)x(t_n) \end{bmatrix} = \begin{bmatrix} R_x(t_1 t_1) \cdots R_x(t_1 t_n) \\ R_x(t_2 t_1) \quad \vdots \\ R_x(t_n t_1) \cdots R_x(t_n t_n) \end{bmatrix}$$

C) FOR STATIONARY PROCESS $(t_N = t_1 + (N-1)\Delta t)$

$$R_x = \begin{bmatrix} R_x(0) & R_x(\Delta t) \cdots R_x[(N-1)\Delta t] \\ R_x(\Delta t) & \vdots \\ R_x[(N-1)\Delta t] & \cdots\cdots R_x(0) \end{bmatrix}$$

SYMETRIC AROUND \ DIAGONAL $(R_x[i\Delta t] = R_x[-i\Delta t])$

D) COVARIANCE MATRIX

1) GENERAL DEFINITION OF ~~RA~~ COVARIANCE BETWEEN TWO RANDOM VARIABLES:

$$E\{[x(t_i) - \bar{x}(t_i)][x(t_j) - \bar{x}(t_j)]\} = \sigma_i \sigma_j \rho_{ij}$$

$\ni \rho_{ij}$ = NORMALIZED COVARIANCE BETWEEN COEFFICIENT OF $x(t_i)$ & $x(t_j)$ (=1 WHEN $i=j$)

2) THE COVARARIANCE MATRIX

a) $\Lambda_x = E[(X-\bar{X})(X^T - \bar{X}^T)]$

b)
$$\Lambda_x = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho_{12} \cdots \sigma_1\sigma_n\rho_{1N} \\ \sigma_2\sigma_1\rho_{21} & \sigma_2^2 \\ \sigma_n\sigma_1\rho_{n1} & \cdots\cdots \sigma_n^2 \end{bmatrix}$$

c) $\Lambda_x = R_x - \bar{X}\bar{X}^T$

d) $\sigma_i^2 = \sigma_j^2 = \sigma^2$; $\rho_{ij} = \rho_{|i-j|}$

$$\Rightarrow \Lambda_x = \sigma^2 \begin{bmatrix} 1 & \rho_1 & \rho_2 \cdots \rho_{n-1} \\ \rho_1 & 1 & \vdots \\ \rho_{n-1} & \cdots & 1 \end{bmatrix}$$

IV) RESPONSE OF LINEAR SYSTEMS TO RANDOM INPUTS

A) ANALYSIS OF THE TIME DOMAIN

1) $X(t) \longrightarrow \boxed{h(t)} \longrightarrow Y(t)$

   a) $Y(t) = \int_0^\infty X(t-\lambda)\, h(\lambda)\, d\lambda$

   b) OR $Y(t) = \int_{-\infty}^{t} X(\lambda)\, h(t-\lambda)\, d\lambda$

2) $h(t) = 0 \ \forall\, t < 0 \ ; \ \int_{-\infty}^{\infty} |h(t)|\, dt < \infty$

B) MEAN AND MEAN SQUARE VALUES OF SYSTEM OUTPUT

1) MEAN VALUE OF OUTPUT: $\overline{Y}$

   $\overline{Y} = E[Y(t)] = E\left[\int_0^\infty X(t-\lambda_1)\, h(\lambda_1)\, d\lambda_1\right] = \overline{X} \underbrace{\int_0^\infty h(\lambda)\, d\lambda}_{\text{d.c. GAIN OF SYSTEM}}$

2) MEAN SQUARE VALUE OF OUTPUT: $\overline{Y^2}$

   $\overline{Y^2} = E[Y^2(t)] = \int_0^\infty d\lambda_1 \int_0^\infty R_X(\lambda_2 - \lambda_1)\, h(\lambda_1)\, h(\lambda_2)\, d\lambda_2$

3) FOR WHITE NOISE $(R_X(\tau) = S_0\, \delta(\tau))$

   a) $\overline{Y^2} = S_0 \int_0^\infty h^2(\lambda)\, d\lambda$

   b) $\overline{Y} = \overline{X}$

C) AUTOCORRELATION FUNCTION OF SYSTEM OUTPUT

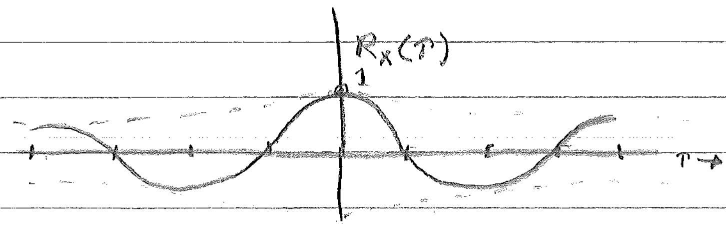
1) $R_Y(\tau) = \int_0^\infty d\lambda_1 \int_0^\infty R_X(\lambda_2 - \lambda_1 - \tau)\, h(\lambda_1)\, h(\lambda_2)\, d\lambda_2$

2) FOR WHITE NOISE:

   $R_Y(\tau) = S_0 \int_0^\infty h(\lambda_1)\, h(\lambda_1 + \tau)\, d\lambda_1$

10-2) $R_X(\tau) = 100 e^{-|\tau|} \cos 10\pi\tau$

a)



b) $\overline{X^2} = R_X(0) = 100$

c) 0 CORRELATION

d) $\rho = -1$

e) 0

10-4) $R_X(\tau) = 16 e^{-|\tau|} \cos \pi\tau + 4 \cos 2\pi\tau$

a) 0

b) EVERY 1 SECOND

c) $R_X(\tau) = 16 e^{-|\tau|} \cos \pi\tau + 4 \cos 2\pi\tau$

$= 16 (\cos|\tau| - j \sin|\tau|) \cos \pi\tau +$ ETC

10-3) $R_X(\tau) = 25 e^{-|\tau|} + 100$

a) $R_X(0) = \overline{X^2} = 125$

b) $\overline{X} = 10$

c) $\sigma^2 = \overline{X^2} - \overline{X}^2 = 25$

10-5) $Z(t) = X(t) - \rho_{XY} Y(t+\tau)$

a) $E[Z^2(t)] = E[X^2(t) - 2X(t)Y(t+\tau) + \rho_{XY}^2 Y^2(t+\tau)]$

$\Rightarrow \sigma_Z^2 = \sigma_X^2 - 2\rho_{XY} R_{XY}(\tau) + \rho_{XY}^2 \sigma_Y^2$

$0 = \frac{d\sigma_X^2}{d\rho_{XY}} = -\cancel{2} R_{XY}(t) + \cancel{2} \rho_{XY} \sigma_Y^2 \Rightarrow \rho_{XY} = \frac{R_{XY}(\tau)}{\sigma^2_Y}$

b)

10-6) $X(t) = A\cos(w_0 t + \theta)$ $\quad$ $Y(t) = B\sin(w_0 t + \theta)$

$\frac{1}{2\pi}\boxed{\theta}$

$R_{XY}(\tau) = \cancel{\int_0^{2\pi} dx \int_0^{2\pi} f(\theta)\, AB\cos(w_0 t + \theta)\sin(w_0 t + \theta)\frac{1}{2\pi}d\theta}$

$$R_{XY}(\tau) = E[X_1 Y_2]$$

$$= \int_0^{2\pi} A\cos(w_0 t + \theta)\, B\sin(w_0 t + \theta + \tau)\,\frac{1}{2\pi}d\theta$$

$$= \int_0^{2\pi} \frac{AB}{2\pi}\cos(w_0 t + \theta)\left[\sin(w_0 t + \theta)\cos\tau + \cos(w_0 t + \theta)\sin\tau\right]d\theta$$

$$= \frac{AB}{2\pi}\int_0^{2\pi}\left\{\cos\tau\,\frac{1}{2}\sin(2w_0 t + 2\theta) + \sin\tau\left[\frac{1}{2} + \frac{1}{2}\cos(2w_0 t + 2\theta)\right]\right\}d\theta$$

$$= \frac{AB}{4\pi}\int_0^{2\pi}\left[\cos\tau\sin(2w_0 t + 2\theta) + \frac{1}{2}\sin\tau + \frac{1}{2}\sin\tau\cos(2w_0 t + 2\theta)\right]d\theta$$

$$= \frac{AB}{4\pi}\left[-\frac{1}{2}\cos\tau\cos(2w_0 t + 2\theta) + \frac{\theta}{2}\sin\tau + \frac{1}{4}\sin\tau\cos(2w_0 t + 2\theta)\right]_0^{2\pi}$$

$$= \frac{AB}{4\pi}\left[\frac{2\pi}{2}\sin\tau\right] = \frac{AB}{4}\sin\tau$$

10-9) $Z(t) = X(t) - Y(t)$ $\quad$ $W(t) = X(t) + Y(t)$

a) $R_Z(\tau) = E\left[(X(t) - Y(t))(X(t+\tau) - Y(t+\tau))\right]$

$$= E\left[X(t)X(t+\tau) - X(t)Y(t+\tau) - Y(t)X(t+\tau) + Y(t)Y(t+\tau)\right]$$

$$= R_X(\tau) - R_{XY}(\tau) - R_{YX}(\tau) + R_Y(\tau)$$

b) $R_W(\tau) = E\left[X(t)X(t+\tau) + X(t)Y(t+\tau) + Y(t)X(t+\tau) + Y(t)Y(\right.$

$$= R_X(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_Y(\tau)$$

$\cancel{c)}$ $R_{ZW}(\tau) = E\left[Z(t)\,W(t+\tau)\right]$

$$= E\left[X(t)X(t+\tau) + X(t)Y(t+\tau) - Y(t)X(t+\tau) - Y(t)Y(t\right.$$

$$= R_X(\tau) + R_{XY}(\tau) - R_{YX}(\tau) - R_Y(\tau)$$

$\cancel{d)}$ $R_{WZ}(\tau) =$

$\cancel{e)}$

CRAM SHEET; TEST 2

I) LAPLACE:

1) $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$ 2) $\mathcal{L}^{-1}\left\{\frac{s+a}{(s+a)^2+\omega_0^2}\right\} = e^{-at}\cos\omega_0 t$

3) $\mathcal{L}^{-1}\left\{\frac{\omega_0}{(s+a)^2+\omega_0^2}\right\} = e^{-at}\sin\omega_0 t$

II) CORRELATION FUNCTIONS

1) $R_x(\tau) = E[x(t)x(t+\tau)]$ (STATIONARY)
2) $\mathcal{R}_x(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt$
3) $\rho = R_x(\tau)/\sigma_x^2$
4) $R_x(0) = \overline{x^2}$; $\lim_{\tau\to\infty} R_x(\tau) = \overline{x}^2$; $R_x(\tau) = R_x(-\tau)$

CROSS-CORRELATION FUNCTIONS

5) $R_{xy}(\tau) = E[x(t)y(t+\tau)]$ (STATIONARY)
6) $R_{xy}(\tau) = R_{yx}(-\tau)$
7) $R_z = R_{x\pm y} = R_x(\tau) + R_y(\tau) \pm R_{xy}(\tau) \pm R_{yx}(\tau)$
8) FOR RANDOM VECTORS: $R_{xy} = E[x(t_1)y(t_1+\tau)]$
9) COVARIANCE MATRIX:

$$\Lambda_x = \begin{bmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{n-1} \\ \rho_1 & 1 & & & \vdots \\ \vdots & & \ddots & & \\ \rho_{n-1} & & & \cdots & 1 \end{bmatrix} \sigma_x^2$$

III) SPECTRAL DENSITY

1) $\overline{x^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)d\omega$
2) $S_x(\omega) = \mathcal{F}\{R_x(\tau)\}$
3) $R_x(0) = \Sigma$ RESIDUES OF POLES INSIDE CURB $C_n < D_n$
4) TAKING DOUBLE LAPLACE FOR L.H.P.

$s \to -s$; TAKE $\mathcal{L}^{-1}$; $t \to -t$

ANTI-LAPLACE; $t \to -t$; $\mathcal{L}^{+1}$; $s \to -s$

5) WHITE NOISE;

$R_x(\tau) = S_0\,\delta(t)$; $S_x(\omega) = S_0$

CRAM SHEET: TEST 3

I) MS & RMS OF OUTPUT

A) $\overline{Y} = E[Y(t)] = \overline{X} \int_0^\infty h(\lambda) d\lambda$

B) $\overline{Y^2} = \int_0^\infty d\lambda_1 \int_0^\infty R_X(\lambda_2 - \lambda_1) h(\lambda_1) h(\lambda_2) d\lambda_2$

$= S_0 \int_0^\infty h^2(\lambda) d\lambda$ FOR WHITE NOISE $[R_X(\tau) = S_0 \delta(t)]$

II) AUTOCORRELATION OF SYSTEM OUTPUT

$R_Y(\tau) = \int_0^\infty d\lambda_1 \int_0^\infty R_X(\lambda_2 - \lambda_1 - \tau) h(\lambda_1) h(\lambda_2) d\lambda_2$

$= S_0 \int_0^\infty h(\lambda_1) h(\lambda_1 + \tau) d\lambda_1$ FOR WHITE NOISE

III) CROSS CORRELATION BETWEEN INPUT AND OUTPUT

$R_{XY}(\tau) = \int_0^\infty R_X(\tau - \lambda) h(\lambda) d\lambda$

$= S_0 h(-\tau)$ FOR WHITE NOISE

IV) SPECTRAL DENSITY OF OUTPUT

$S_Y(\omega) = S_X(\omega) |H(j\omega)|^2 = S_X(s) H(s) H(-s)$

V) BAYES RULE AND SUCH

$f_r[r/m_i] P[m_i] \geq f_r[r/m_j] P[m_j]$

$P(c) = \sum_{\text{ALL } j} P[c/r_j] P[r_j]$

VI) APOSTERIORI PROB. OF $m_i = P[m_i/r_j]$